The symplectic normal space of a Lagrangian fibration

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Contents

Contents

Group actions

Let $\psi : G \times M \longrightarrow M$ be an (left) action

Notation:

- $G \cdot z$ orbit of $z \in M$
- Infinitesimal generator $\xi_M \in \mathfrak{X}(M)$

$$\xi_M(z) = T_e\psi_z(\xi), \quad \xi \in \mathfrak{g}$$

- $\mathfrak{g} \cdot z$ tangent space of the orbit
- G_z isotropy subgroup
- $\mathfrak{g}_z$ Lie algebra of G_z

Twisted products

Assume that a subgroup $H \subset G$ acts on a manifold A:

$$H \times A \longrightarrow A$$

On $G \times A$ there is a twisting action of H:

$$h \cdot^T (g, a) = (gh^{-1}, h \cdot a), \quad h \in H$$

Proposition

If H acts on properly on A:

- *the twisting action is free and proper. The quotient space $(G \times A)/H^T$ is a manifold, the twisted product, denoted as $G \times_H A$ and its elements $[g, a]_H \quad g \in G, \ a \in A$.*
- *the twisted product has a proper G action given by*

$$g \cdot [g', a]_H = [gg', a]_H$$

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Palais' Theorem

Theorem (Tube theorem for G-spaces)

Fix a point $z \in M$. Assume M has a G_z-invariant metric (always available by compactness of G_z).

Define $\mathbf{S} = (\mathfrak{g} \cdot z)^{\perp} \subset T_z M$ then:

$$\begin{aligned} s : G \times_{G_z} \mathbf{S} &\longrightarrow U \subset M \\ [g, a]_{G_z} &\longmapsto g \cdot Exp(a) \end{aligned}$$

is G-equivariant diffeomorphism onto a G-invariant open neighborhood U of $G \cdot z$ satisfying $s([e, 0]_{G_z}) = z$.

This result provides normal form for a neighborhood of $G \cdot z$. It is a semi-local model: it is global with respect to G-directions but local with respect to transverse ones.

Contents

Hamiltonian G-spaces

A Hamiltonian G-space is $(M, \omega, G, \mathbf{J})$, where

- (M, ω) is a symplectic manifold,
- G is a Lie group which acts on M properly and symplectically,
- $\mathbf{J} : M \to \mathfrak{g}^*$ is a momentum map associated to the action, i.e.

$$i_{\xi_M}\omega = d\langle \mathbf{J}(\cdot), \xi\rangle \quad \forall \xi \in \mathfrak{g}$$

We will not assume that $\mathbf{J}$ is Ad^*-equivariant.

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Cotangent lifted actions

$$(T^*Q, \Omega_Q, G, \mathbf{J}_Q)$$

Ω_Q is the canonical symplectic form on T^*Q

Action:

$$\psi : G \times Q \to Q \text{ proper action}$$

$$\Downarrow$$

$$\text{cotangent lift } T^*\psi : G \times T^*Q \to T^*Q$$

$$T^*\psi(g, \alpha_q) = T_q^*\psi_{g^{-1}}(\alpha_q)$$

The momentum map $\mathbf{J}_Q : T^*Q \to \mathfrak{g}^*$

$$\langle \mathbf{J}_Q(\alpha_q), \xi \rangle = \alpha_q(\xi_Q(q))$$

In this case $\mathbf{J}_Q$ is equivariant.

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Magnetic cotangent lifted actions

$$(T^*Q, \Omega_Q - \pi^*\beta, G, \mathbf{J})$$

$\pi : T^*Q \to Q$ is the canonical projection

β is a G-invariant closed 2-form on Q

Assume that there exists $\phi : Q \to \mathfrak{g}^*$ such that

$$\beta(\xi_Q, \cdot) = \langle d\phi, \xi \rangle \ \xi \in \mathfrak{g}$$

The momentum map is $\mathbf{J} = \mathbf{J}_Q - \phi \circ \pi$ (in general, non-equivariant)

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Coadjoint group orbits

$$(\mathcal{O}_\mu, \omega_\mu, G, \mathbf{J}_\mu)$$

G Lie group and $\mu \in \mathfrak{g}^*$

$\mathcal{O}_\mu = G \cdot \mu \subset \mathfrak{g}^*$ is the coadjoint orbit

ω_μ is the Kirillov-Kostant-Souriau form on $\mathcal{O}_\mu$

$$\omega_\mu(\nu)(\xi_{\mathfrak{g}^*}(\nu), \bar{\xi}_{\mathfrak{g}^*}(\nu)) = -\nu[\xi, \bar{\xi}]$$

The momentum map is $\mathbf{J}_\mu(\nu) = -\nu$

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H-restricted Coadjoint group orbits

$$(\mathcal{O}_\mu, \omega_\mu, H, \mathbf{J}_H)$$

H Lie subgroup of G and $\mu \in \mathfrak{g}^*$

$\mathcal{O}_\mu = G \cdot \mu$ is the coadjoint orbit

ω_μ is the Kirillov-Kostant-Souriau form on $\mathcal{O}_\mu$

$$\omega_\mu(\nu)(\xi_{\mathfrak{g}^*}(\nu), \bar{\xi}_{\mathfrak{g}^*}(\nu)) = -\nu[\xi, \bar{\xi}]$$

The momentum map is $\mathbf{J}_H(\nu) = -\nu_{|\mathfrak{h}} \in \mathfrak{h}^*$

Contents

Symplectic normal spaces

Let $(M, \omega, G, \mathbf{J})$ be a Hamiltonian G-space. Given $z \in M$ and $\mathbf{J}(z) = \mu \in \mathfrak{g}^*$.

Definition

A complement N of $\mathfrak{g}_\mu \cdot z$ inside $T_z\mathbf{J}^{-1}(\mu)$ is a symplectic normal space at z. N is a symplectic vector space.

Proposition

Assume G acts freely on M, then $\pi : M \longrightarrow M/G_\mu$ is smooth.

Given $z \in M$ with $\mathbf{J}(z) = \mu$ and N is a symplectic normal space at z, there is a symplectomorphism:

$$T_{\pi(z)}(\mathbf{J}^{-1}(\mu)/G_\mu) \cong N$$

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Marle-Guillemin-Sternberg model

Assume (V, ω_V) is a linear symplectic space with a $K \subset G_\mu$ proper Hamiltonian action. Let $\mathfrak{m}$ such that: $\mathfrak{g}_\mu = \mathfrak{k} \oplus \mathfrak{m}$. The twisted product:

$$Y = G \times_K (\mathfrak{m}^* \times V)$$

is a symplectic manifold and has a proper Hamiltonian G-action with momentum map:

$$\begin{aligned} \mathbf{J}_Y : Y = G \times_K (\mathfrak{m}^* \times V) &\longrightarrow \mathfrak{g}^* \\ [g, \nu, v]_K &\longrightarrow \mathrm{Ad}^*_{g^{-1}}(\mu + \nu + \mathbf{J}_V(v)) \end{aligned}$$

Theorem (Hamiltonian Tube Theorem)

Let (M, ω) be a Hamiltonian G-space. Fix $z \in M$, consider N a symplectic normal space at z: There is a G-equivariant symplectomorphism:

$$\mathcal{T} : (Y, \Omega_Y) \longrightarrow (U, \omega) \subset (M, \omega)$$

where:

$$Y = G \times_{G_z} (\mathfrak{m}^* \times N) \text{ and } \mathfrak{g}_\mu = \mathfrak{g}_z \oplus \mathfrak{m}$$

satisfying $\mathcal{T}([e, 0, 0]_{G_z}) = z$.

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Symplectic normal space for cotangent lifted actions

Theorem (Cotangent Normal Slice (free action))

Assume G acts freely on Q.
Let $z \in T_q^* Q$ with $\mathbf{J}(z) = \mu$.
Define $\mathbf{S} = (\mathfrak{g} \cdot q)^{\perp}$.
The symplectic normal space N satisfies:

$$N \cong T_\mu \mathcal{O}_\mu \oplus \mathbf{S} \oplus \mathbf{S}^*$$

with symplectic form

$$\Omega_N = \begin{bmatrix} \omega_\mu & 0 & 0 \\ 0 & 0 & Id \\ 0 & -Id & 0 \end{bmatrix}$$

where $\omega_\mu(\xi_{\mathfrak{g}^*}(\mu), \overline{\xi}_{\mathfrak{g}^*}(\mu)) = -\langle \mu, [\xi, \overline{\xi}] \rangle$ is the KKS form of $\mathcal{O}_\mu$.

Symplectic normal space for cotangent lifted actions

Theorem (Cotangent Normal Slice)

Let $z \in T_q^*Q$ with $\mathbf{J}(z) = \mu$, denote $\alpha = z|_{\mathbf{S}}$ and $H = G_q$. Define

- $B = (\mathfrak{g}_{q,\mu} \cdot \alpha)^\circ \subset \mathbf{S} = (\mathfrak{g} \cdot q)^\perp$
- $\mathfrak{n}$ *be the symplectic normal space at μ for the H action on $\mathcal{O}_\mu$.*

Then the symplectic normal space N satisfies:

$$N \cong \mathfrak{n} \oplus B \oplus B^*$$

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where $\omega_\mu(\xi, \bar{\xi}) = -\langle \mu, [\xi, \bar{\xi}] \rangle$.

M. Perlmutter, M. Rodríguez-Olmos, M.E. Sousa-Dias: *The symplectic normal space of a cotangent-lifted action.* Differential Geom. Appl., **26(3)**, 277–297 (2008).

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Lagragian fibrations

Consider:

- $(L, \omega, G, \mathbf{J})$ be a Hamiltonian G-space
- Q a smooth manifold with a proper G-action
- $\pi : L \to Q$ a fiber bundle satisfying:
  - $\forall q \in Q$: $\pi^{-1}(q)$ is a Lagrangian submanifold of L
  - $\forall z \in L, \forall g \in G$: $\pi(g \cdot z) = g \cdot \pi(z)$ (equivariance)

$(L, \omega, G, \mathbf{J}, \pi : L \to Q)$ will be called a Lagrangian fibration.

Example: Any magnetic cotangent lifted action because the magnetic term vanishes along the fibers of $T^*Q \to Q$.

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Affine coadjoint orbit

Given a group cocycle $\sigma : G \to \mathfrak{g}^*$ consider

$$g \cdot \nu = Ad^*_{g^{-1}}\nu + \sigma(g)$$

this action is called affine coadjoint action.

$\widetilde{\mathcal{O}}_\mu = G \cdot \mu$ is the affine coadjoint orbit, which is also a symplectic manifold with the following symplectic structure $\widetilde{\omega}_\mu$

$$\widetilde{\omega}_\mu(\nu)(\xi_{\mathfrak{g}^*}(\nu), \bar{\xi}_{\mathfrak{g}^*}(\nu)) = -\langle \nu, [\xi, \bar{\xi}] \rangle + \Sigma(\xi, \bar{\xi})$$

where $\Sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is given by

$$\Sigma(\xi, \bar{\xi}) := \langle T_e\sigma(\xi), \bar{\xi} \rangle$$

The momentum map is $\widetilde{\mathbf{J}}_\mu(\nu) = -\nu$

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$$\sigma(g) = \mathbf{J}(g \cdot z) - Ad^*_{g^{-1}}(\mathbf{J}(z))$$

Theorem

Let $z \in L$ with $\mathbf{J}(z) = \mu$, denote $\pi(z) = q$ and $H = G_q$. Define

- $B = ((\widetilde{\mathfrak{g}}_\mu \cap \mathfrak{g}_q) \cdot z)^\circ \subset \mathbf{S} = (\mathfrak{g} \cdot q)^\perp$
- $\widetilde{\mathfrak{n}}$ *be the symplectic normal space at* μ *for the H-affine coadjoint action on* $\widetilde{\mathcal{O}}_\mu$ *respect to the G-cocycle* σ.

Then the symplectic normal space N satisfies:

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with symplectic form

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Thank you for your attention!