Conformal Killing vector fields and virial theorems[1]

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[1] Joint work with José F. Cariñena, Eduardo Martínez and Irina Gheorghiu

Outline – Conformal Killing v.f.s and virial theorems

§1 Historical account and Motivation

In 1870 Clausius introduced the Virial Theorem in statistical mechanics [2]. In the original formulation, the VT establishes for a one-particle system a relation between the time averages of the kinetic energy and of the scalar product of trajectory by force,

$$\langle\langle T\rangle\rangle = -\frac{1}{2}\langle\langle \mathbf{r}\cdot\mathbf{F}\rangle\rangle; \tag{1}$$

the time average means

$$\langle\langle G\rangle\rangle = \lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^{\tau} G(\gamma(t))\,dt, \tag{2}$$

where γ is the evolution curve.

In the case of a conservative force $\mathbf{F} = -\nabla V$, he obtained $\langle\langle T\rangle\rangle = \frac{1}{2}\langle\langle \mathbf{r}\cdot\nabla V\rangle\rangle$.

[2] R. Clausius, *XVI. On a mechanical theorem applicable to heat.* Philosophical Magazine Series 4, **40**(265):122-127, 1870.

§1 Historical account and Motivation

The Virial Theorem (VT) have a wide range of applications from dynamical and thermodynamical systems, to dust and gas of interstellar space and cosmology [3], and it is the main reason to think that dark matter exist [4].

The modern approach to the VT[5] uses the Hamiltonian formalism and establishes, under some general conditions, that the time average of the Poisson bracket $\{\cdot,\cdot\}$ of an observable G with the Hamiltonian H vanishes, i.e. $\langle\langle\{G,H\}\rangle\rangle = 0$.

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[3] G.W. Collins,*The Virial Theorem in Stellar Astrophysics*. Astronomy and Astrophysics Series **7**, Tucson, AZ: Pachart Pub. House, 1978.

[4] J. An and N.W. Evans, *Modified virial formula*e *and the theory of mass estimators*. Mon. Not. R. Astron. Soc., 413, 1744-1752, 2011.

[5] J.F.Cariñena, F.Falceto and M.F.Rañada, *A geometric approach to a generalized virial theorem*. J. Phys. A: Math. Theor. **45**, 395210, 2012.

§1 Historical account and Motivation

Ghori[6] wrote in '98 the Virial Theorem (VT) in the setting of Poincaré's formalism, i.e. he wrote the above result in quasi-coordinates (q^i, π_i) on the phase space T^*M.

Inspired in the work of Ghori and using a similar approach, Cariñena *et al.*[7] wrote the VT in Boltzmann formalism (i.e. the result is written in quasi-coordinates (q^i, w^i) on the tangent bundle $\tau_M : TM \to M$) and generalised the theorem in the framework of dynamics on Lie algebroids $(A, [\cdot, \cdot]_A, \rho)$.

In the present work we try to explore in the context of Lagrangian Mechanics the information that we can extract from virial functions (here considered as smooth bounded function on M) associated to conformal Killing v.f.s.

[6] Q.K. Ghori, *Note on the Virial theorem.* Acta Mecanica Sinica **14**, 76–77, 1998.

[7] J.F. Cariñena, I. Gheorghiu, E. Martínez and Patrícia Santos, *Virial theorem in quasi-coordinates and Lie algebroid formalism*, Int. J. Geom. Methods Mod. Phys. **11**, 1450055, 2014.

§2 Virial Theorem for mechanical Lagrangian systems

2.1 Mechanical Lagrangian systems

→ (M, g) is a (pseudo-)Riemann manifold, i.e. g is a (non-degenerate) positive definite symmetric 2-covariant tensor field on M.

→ $T_g \in C^\infty(TM)$ is the kinetic energy defined by the metric g, given by

$$T_g(q, v) = \frac{1}{2} g_q(v, v), \text{ for all } (q, v) \in TM. \tag{3}$$

We define a Lagrangian of mechanical type on (M, g), $L \in C^\infty(TM)$, as follows:

$$\begin{aligned} L(q, v) &= T_g(q, v) - \widetilde{V}(q, v) \qquad (4) \\ &= \frac{1}{2} g_q(v, v) - V(q), \end{aligned}$$

for all $(q, v) \in TM$, where $\tau_M : TM \to M$ is the tangent bundle projection and the basic function $V \equiv \widetilde{V} = \tau_M^* V \in C^\infty(TM)$ is the potential energy of the system.

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2.1 Mechanical Lagrangian systems

Consider:

- $\rightarrow$ $(U, q^1, \ldots, q^n)$ a local chart on the manifold M;
- $\rightarrow$ $\{\partial/\partial q^j \mid j = 1, \ldots, n\}$ the coordinate basis of $\mathfrak{X}(U)$.

Then a vector in $q \in U$ is given by $v = v^j \, (\partial/\partial q^j)_q$, with $v^j = \langle dq^j, v \rangle$ being the usual velocities. In coordinates the Lagrangian is given by

$$L(q, v) = \frac{1}{2} g_{ij}(q) v^i v^j - V(q), \qquad (5)$$

where summation is understood for repeated indices.

§2 Virial Theorem for mechanical Lagrangian systems

2.1 Mechanical Lagrangian systems

The system defined by the regular Lagrangian $L = T_g - V$ is a Hamiltonian system $(TM, \{\cdot,\cdot\}, H)$. The Hamiltonian H is the energy of the Lagrangian system given by

$$E_L = \Delta L - L = T_g + V \quad (6)$$ [8]

and the Poisson bracket $\{\cdot,\cdot\}$ is defined by the symplectic structure (Cartan 2-form) $\omega_L = -d\theta_L = -d(dL \circ S)$ [9] as follows:

$$\{F, G\} := \omega_L(X_F, X_G), \quad (7)$$

for all $F, G \in C^\infty(TM)$, where the Hamiltonian v.f. X_F is determined by the dynamical equation $i(X_F)\omega_L = dF$.

[8] $\Delta \in \mathcal{X}(TM)$ is the Liouville dilation v.f. determined by $\Delta f(q,v) = \frac{d}{dt} f(q, e^t v)\big|_{t=0}$ for all $f \in C^\infty(TM)$. In coordinates we have $\Delta = v^i \partial_{v^i}$.

[9] $S : T(TM) \to T(TM)$ is the vertical endomorphism determined by $S(\mathcal{X}) f(q,v) = \frac{d}{dt} f(q, v + tX)\big|_{t=0}$, where $X = T_{(q,v)}\tau_M \mathcal{X}$, for all $f \in C^\infty(TM)$ and $\mathcal{X} \in \mathfrak{X}(TM)$. In coordinates we have $S = \partial_{v^i} \otimes dq^i$.

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2.1 Mechanical Lagrangian systems

In coordinates, the symplectic structure is given by

$$\omega_L = g_{ij}\, dq^i \wedge dv^j + \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial q^k} v^j - \frac{\partial g_{kj}}{\partial q^i} v^j\right) dq^i \wedge dq^k, \tag{8}$$

and the solution of the dynamical equation $i_{\Gamma_L}\omega_L = dE_L$ turns out to be

$$\Gamma_L(q,v) = v^i \frac{\partial}{\partial q^i} - \left(\Gamma^i_{jk}(q) v^j v^k + g^{ij}(q)\frac{\partial V}{\partial q^j}(q)\right)\frac{\partial}{\partial v^i}, \tag{9}$$

where $\Gamma^i_{jk} = \frac{1}{2} g^{il}\left(\frac{\partial g_{lj}}{\partial q^k} + \frac{\partial g_{lk}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l}\right)$ are the Christoffel symbols w.r.t. the Levi-Civita connection defined by the metric g, and g^{ij} are the inverse matrix entries of the Riemann structure g.

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§2 Virial Theorem for mechanical Lagrangian systems

2.2 Coordinates expression and complete lifts

The Virial Theorem (VT) states for a smooth bounded function G on M that the time average of the Poisson bracket $\dot{G} = \{G, E_L\} = \Gamma_L(G) = -X_G(E_L)$ vanish:

$$\langle\!\langle \{G, E_L\} \rangle\!\rangle = 0. \tag{10}$$

In coordinates:

$$\left\langle\!\!\left\langle \frac{\partial G}{\partial q^i} v^i - \frac{\partial G}{\partial v^i} \left(\Gamma^i_{jk} v^j v^k + g^{ij} \frac{\partial V}{\partial q^j} \right) \right\rangle\!\!\right\rangle = 0. \tag{11}$$

§2 Virial Theorem for mechanical Lagrangian systems

2.2 Coordinates expression and complete lifts

Let $X = X^i \partial_{q^i}$ be a v.f. on M and $X^c = X^i \partial_{q^i} + \frac{\partial X^i}{\partial q^k} v^k \partial_{v^i}$ its complete lift on TM. If the virial function is defined by $G = \langle \theta_L, X^c \rangle = X^i \frac{\partial L}{\partial v^i}$ then

$$\Gamma_L(G) = X^c(L). \tag{12}$$

In fact, since $\mathcal{L}_{\Gamma_L} \theta_L = dL$ we have

$$\langle dL, X^c \rangle = \langle \mathcal{L}_{\Gamma_L} \theta_L, X^c \rangle = i(X^c) \mathcal{L}_{\Gamma_L} \theta_L = \mathcal{L}_{\Gamma_L} i(X^c) \theta_L + i([X^c, \Gamma_L]) \theta_L = \mathcal{L}_{\Gamma_L} G.$$

Therefore, the Virial Theorem reduces to

$$\langle\langle X^c(L) \rangle\rangle = 0, \tag{13}$$

that is,

$$\left\langle\!\!\left\langle X^i \frac{\partial L}{\partial q^i} + \frac{\partial X^i}{\partial q^k} v^k \frac{\partial L}{\partial v^i} \right\rangle\!\!\right\rangle = 0, \tag{14}$$

Thus, for mechanical systems the VT states that $\langle\langle X^c(T_g) - X(V) \rangle\rangle = 0$.

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2.2 Coordinates expression and complete lifts

One of the more important properties of complete lifts is the following relation:

$$\mathcal{L}_{X^c} T_g = T_{\mathcal{L}_X g}. \tag{15}$$

In fact, for all $v \in TM$,

$$\begin{aligned} \mathcal{L}_{X^c} T_g(v) &= \frac{d}{dt} T_g \circ T\phi_t(v)\Big|_{t=0} = \frac{d}{dt}\frac{1}{2} g(T\phi_t(v), T\phi_t(v))\Big|_{t=0} \\ &= \frac{1}{2}(\mathcal{L}_X g)(v,v) = T_{\mathcal{L}_X g}(v,v). \end{aligned}$$

Hence, $X \in \mathfrak{X}(M)$ is a Killing vector filed w.r.t. the Riemann structure g (i.e. $\mathcal{L}_X g = 0$) iff $X^c \in \mathfrak{X}(TM)$ is a symmetry for the corresponding free Lagrangian T_g.

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Proposition

$X \in \mathfrak{X}(M)$ is a Killing vector field w.r.t. the Riemann structure g iff $X_G = X^c$, where $G = \langle \theta_L, X^c \rangle$ is the associated function.

If $X \in \mathfrak{X}(M)$ is a Killing v.f. for g, the Virial Theorem reduces to

$$\langle\langle \mathcal{L}_X V \rangle\rangle = 0, \tag{16}$$

because $\Gamma_L(G) = X^c(L) = X^c(T_g - V) = -X(V)$. From this result we can extract some information about the potential energy, as we can see in the following example.

§2 Virial Theorem for mechanical Lagrangian systems

2.3 Example - Periodic Toda lattice with n particles

A periodic Toda lattice system with n particles (each particle as mass m) is determined by a Lagrangian on $T\mathbb{R}^n$ defined by

$$L(q,v) = T(q,v) - V(q) = \frac{1}{2}\sum_{i=1}^{n} m\, v_i^2 - \sum_{i=1}^{n} e^{q_i - q_{i+1}},$$

where $q_{n+1} = q_1$.

The vector field $X_k = \partial_{q^k}$, for a fixed $k = 1, \ldots, n$, is a Killing vector w.r.t. the Euclidean metric on $\mathbb{R}^n$. In this case, the Virial Theorem implies that

$$0 = \langle\langle \mathcal{L}_{X_k} V \rangle\rangle = \langle\langle e^{q_k - q_{k+1}} - e^{q_{k-1} - q_k} \rangle\rangle \iff \langle\langle e^{q_k - q_{k+1}} \rangle\rangle = \langle\langle e^{q_{k-1} - q_k} \rangle\rangle$$

Hence, $\langle\langle V \rangle\rangle = n \langle\langle e^{q_1 - q_2} \rangle\rangle$.

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Hence, $\langle\!\langle V \rangle\!\rangle = n \langle\!\langle e^{q_1 - q_2} \rangle\!\rangle$.

§3 Virial Theorem and conformal Killing v.f.s

Conformal Killing v.f.s and in particular homothetic v.f.s have been relevant in many problems in physics [10], we now explore the information that we can extract from them in the context of the Virial Theorem.

Recall that $X \in \mathfrak{X}(M)$ is either a conformal or a homothetic v.f. when its flow ϕ_t is made of conformal maps or homothethies:

$$\text{conformal vector field}: \quad \mathcal{L}_X g = f\, g, \quad f \in C^\infty(M),$$
$$\text{homothetic vector field}: \quad \mathcal{L}_X g = \lambda\, g, \quad \lambda \in \mathbb{R}.$$

Proper conformal vector fields are those v.f.s for which the conformal factor f is non constant and similarly a proper homothetic v.f. is when $\lambda \neq 0$.

[10] S.D. Maharaj, R. Maartens and M.S. Maharaj, *Conformal symmetries in static spherically symmetric spacetimes*, Int. J. Theor. Phys. **34**, 2285–2291 (1995)

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§3 Virial Theorem and conformal Killing v.f.s

3.1 Functions linear in the velocities

If G is linear in the velocities there exists a 1-form $\alpha = \alpha_j(q)\, dq^j$ on M, such that,

$$G = \widehat{\alpha} = \alpha_j(q)\, \widehat{dq^j} = \alpha_j(q) v^j, \tag{17}$$

and $X = \widehat{g}^{-1}(\alpha) = \alpha_j\, g^{ji}\, \partial_{q^i}$ is the related v.f. to G, where $\widehat{g} : \mathfrak{X}(M) \to \Omega^1(M)$ is a regular map defined by $\langle \widehat{g}(X), Y \rangle = g(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$.

If $X = X^i \partial_{q^i}$ is the above v.f. then the smooth function $G = \widehat{\alpha}$ is nothing but

$$G = \langle \theta_L, X^c \rangle. \tag{18}$$

In coordinates, the above relation is easy to check:

$$\langle \theta_L, X^c \rangle = X^i \frac{\partial L}{\partial v^i} = X^i g_{ik} v^k = \alpha_j\, g^{ji} g_{ik} v^k = \widehat{\alpha}.$$

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§3 Virial Theorem and conformal Killing v.f.s

3.2 Virial functions associated to conformal Killing v.f.s

Concerning Hamiltonian v.f. and complete lifts, we can prove the following:
$X \in \mathfrak{X}(M)$ is a conformal Killing v.f. for g with conformal factor $f \in C^\infty(M)$ iff $X_{\widehat{\alpha}} = X^c - f\,\Delta$, where α is the 1-form $\alpha = \widehat{g}(X)$.

Theorem (Virial Theorem for conformal Killing v.f.s)

Consider a Lagrangian of mechanical type $L = T_g - V$, a conformal Killing v.f. X for g with conformal factor $f \in C^\infty(M)$, and the related 1-form $\alpha = \widehat{g}(X)$. If $G = \widehat{\alpha} = \langle \theta_L, X^c \rangle$ is bounded then

$$\langle\langle fT_g - \mathcal{L}_X V \rangle\rangle = 0. \tag{19}$$

Note that, when X is a homothetic v.f. (i.e. $\mathcal{L}_X g = \lambda g$ with $\lambda \neq 0$) and V is a X-homogenous of degree ν (i.e. $\mathcal{L}_X V = \nu V$), the VT reduces to $\langle\langle \lambda T_g - \nu V \rangle\rangle = 0$.

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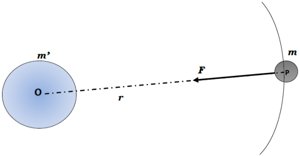
§3 Virial Theorem and conformal Killing v.f.s

3.3 Example - Kepler problem

Consider a particle P of mass m moving in a plane under the action of a central force $F(r) = -\gamma mm'/r^2$ on the direction of a fixed point O of mass $m' \gg m$, where $\gamma > 0$ and $r = \text{dist}(O, P)$.



Let θ be the angle that the line OP makes with a fixed direction on the plane. Then the dynamics is determined by $L = T - V = \frac{m}{2}(v_r^2 + r^2 v_\theta^2) + \frac{\gamma mm'}{r}$.

The infinitesimal generator of dilations $X = r\partial_r$ is a 2-homothetic v.f. for the Riemann metric and V is a X-homogeneous function of degree -1. Applying the VT for conformal killing v.f.s, we obtain the classical result:

$$\langle\langle 2T + V \rangle\rangle = 0 \Longleftrightarrow \langle\langle T \rangle\rangle = -\frac{1}{2}\langle\langle V \rangle\rangle.$$

§3 Virial Theorem and conformal Killing v.f.s

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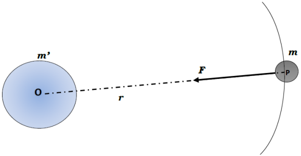

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§3 Virial Theorem and conformal Killing v.f.s

3.3 Example - Spherical geometry

Consider the motion of a unity mass point on a sphere of radius R centered at the origin and the usual spherical polar coordinates, i.e. a point P on the sphere is fixed by two coordinates (θ, ϕ) such that

$$\mathbf{x}(\theta, \phi) = (R\, \sin\theta\, \cos\phi, R\, \sin\theta\, \sin\phi, R\, \cos\theta),$$

The arc-length is given by

$$ds^2 = R^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

and the kinetic energy defined by the metric is

$$T = \frac{1}{2}R^2(v_\theta^2 + \sin^2\theta\, v_\phi^2).$$

Suppose that the motion is under the action of a potential function V.

§3 Virial Theorem and conformal Killing v.f.s

3.3 Example - Spherical geometry

The v.f. $X = \sin\theta\,\partial_\theta$, with complete lift

$$X^c = \sin\theta\,\frac{\partial}{\partial\theta} + \cos\theta\, v_\theta\,\frac{\partial}{\partial v_\theta},$$

is a conformal Killing v.f. for the metric, with conformal factor $f(\theta) = 2\cos\theta$.

In this case, the Virial Theorem for conformal Killing v.f.s implies that

$$\left\langle\!\!\left\langle 2\,\cos\theta\, T_g - \sin\theta\,\frac{\partial V}{\partial\theta}\right\rangle\!\!\right\rangle = 0,$$

that is,

$$\left\langle\!\!\left\langle \cos\theta\, R^2(v_\theta^2 + \sin^2\theta\, v_\phi^2) - \sin\theta\,\frac{\partial V}{\partial\theta}\right\rangle\!\!\right\rangle = 0.$$

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§3 Virial Theorem and conformal Killing v.f.s

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