Symplectic Harmonicity and Generalized Coeffective Cohomologies

Raquel Villacampa

Joint work with Luis Ugarte (arXiv:1404.6777(math.SG))

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PRINCIPAL IDEAS

1. There exist several cohomologies on symplectic manifolds: harmonic, primitive, filtered and coeffective. All of them appear independently in the literature.

   Objective 1: Relate all these cohomologies by defining the generalized coeffective cohomology.

2. Harmonic, primitive and filtered cohomologies are defined in terms of $d^{\wedge}$, ∂_+, ∂_- and therefore these cohomologies are difficult to compute.

   Objective 2: Provide a coeffective version of these cohomologies so that computations are easier.

3. The dimension of all these cohomology groups can vary when the symplectic form does (notion of *flexibility*).

   Objective 3: Study harmonic and filtered flexibility in terms of coeffective flexibility.

SYMPLECTIC HARMONICITY I

- (M^{2n}, ω) symplectic manifold.
- $\Omega^*(M)$ the space of differential forms on M.

[Brylinski 88] $\alpha \in \Omega^*(M)$ is symplectically harmonic if $d\alpha = 0 = d^\wedge \alpha$. ($d^\wedge$ is the adjoint of d with respect to $*_\omega$.)

- Notation: $\Omega^*_{\mathrm{hr}}(M)$.
- Problem: there are non-zero exact symplectically harmonic forms (important difference w.r.t. the usual Hodge Theory).

Symplectically harmonic cohomology: $H^q_{\mathrm{hr}}(M) = \dfrac{\Omega^q_{\mathrm{hr}}(M)}{(\Omega^q_{\mathrm{hr}}(M) \cap \operatorname{im} d)}$.

- $H^q_{\mathrm{hr}}(M) \overset{?}{=} H^q(M)$, $q = 0, \dots, 2n$.

$$\text{Yes} \Longleftrightarrow \text{HLC } (L^k : H^{n-k} \xrightarrow{\cong} H^{n+k},\ k = 1, \dots, n).$$

[Mathieu 95], [Yan 96]

Symplectic harmonicity II

Theorem [Ibáñez-Rudyak-Tralle-Ugarte 01], [Yamada 02]

- $H^q_{\mathrm{hr}}(M) = P^q(M,\omega) + L(H^{q-2}_{\mathrm{hr}}(M)),\ \ q = 0, \ldots, n,$
- $H^q_{\mathrm{hr}}(M) = \mathrm{Im}\,\{L^{q-n} : H^{2n-q}_{\mathrm{hr}}(M) \to H^q(M)),\ \ q = n+1, \ldots, 2n,$

where

$$P^q(M) = \{[\alpha] \in H^q(M) \,|\, L^{n-q+1}[\alpha] = 0\} \subset H^q_{\mathrm{hr}}(M).$$

If (M^{2n},ω) is of finite type: $h_q(M) = \dim H^q_{\mathrm{hr}}(M)$ is finite.

- $h_i = b_i$ for $i = 0, 1, 2$.
- If (M^{2n},ω) is closed: $h_{2n} = b_{2n},\ \ h_{2n-1}$ is even.

PRIMITIVE COHOMOLOGIES [Tseng-Yau I, 12], [Tseng-Yau II, 12]

New finite dimensional cohomologies on symplectic manifolds that contain unique harmonic representative within each class:

- Consider d and $d^\wedge$.

$$H^*_{d+d^\wedge} = \frac{\text{Ker } (d + d^\wedge)}{\text{Im } dd^\wedge}, \quad H^*_{dd^\wedge} = \frac{\text{Ker } dd^\wedge}{\text{Im } d + \text{Im } d^\wedge}.$$

  (type Bott-Chern and Aeppli)

  Primitive cohomologies: $PH^q_{d+d^\wedge}$ and $PH^q_{dd^\wedge}$, for $q \leq n$.

- Idea: Complex case $d = \partial + \bar{\partial}$.

  Symplectic case: $d = \partial_+ + \omega \wedge \partial_- \Longrightarrow H^*_{\partial_+}$ and $H^*_{\partial_-}$.

  Primitive cohomologies: $PH^q_{\partial_+}$ and $PH^q_{\partial_-}$, for $q \leq n - 1$.

FILTERED COHOMOLOGIES [Tsai-Tseng-Yau III, 14]

$$\{PH^*_{d+d^\wedge},\ PH^*_{dd^\wedge},\ PH^*_{\partial_+},\ PH^*_{\partial_-}\}$$ are primitive cohomologies.

Idea: There exist non-primitive ones?

Filtered cohomologies: $F^pH^q_+,\ F^pH^q_-$, for $q = 0, \ldots, n+p$, and $p = 0, \ldots, n$, that extend the primitive ones.

Relations between cohomologies for closed symplectic manifolds:

- $PH^q_{\partial_+} = PH^q_{\partial_-}$, $q \leq n-1$.
- $PH^q_{dd^\wedge} = PH^q_{d+d^\wedge}$, $q \leq n$.
- $F^pH^{n+p}_+ = PH^{n-p}_{dd^\wedge}$.
- $F^pH^{n+p}_- = PH^{n-p}_{d+d^\wedge}$.
- $F^pH^q_+ = F^pH^q_-$.

Coeffective cohomology I

Definition [Bouché 90]

$\alpha \in \Omega^*(M)$ is coeffective if $L_\omega(\alpha) := \alpha \wedge \omega = 0$. Notation: $\mathfrak{C}^q_{(1)}(M)$.

- $d(\mathfrak{C}^q_{(1)}(M)) \subset \mathfrak{C}^{q+1}_{(1)}(M)$.

Coeffective cohomology: $$H^q_{(1)}(M) = \frac{\mathrm{Ker}\,\{d : \mathfrak{C}^q_{(1)}(M) \to \mathfrak{C}^{q+1}_{(1)}(M)\}}{\mathrm{Im}\,\{d : \mathfrak{C}^{q-1}_{(1)}(M) \to \mathfrak{C}^q_{(1)}(M)\}}.$$

- $L_\omega : \Omega^q \to \Omega^{q+2}$ injective $\forall q \leq n-1 \Longrightarrow H^q_{(1)}(M) = 0,\ \forall q \leq n-1.$
- $H^q_{(1)}(M) = H^q(M),\ q = 2n.$

Coeffective cohomology II

If (M^{2n}, ω) is of finite type:

$$\underbrace{H^0_{(1)}(M),\ \ldots, H^{n-1}_{(1)}(M)}_{=0},\ H^n_{(1)}(M),\ \underbrace{H^{n+1}_{(1)}(M),\ \ldots, H^{2n}_{(1)}(M)}_{\text{are finite dimensional}}.$$

Notation: $c^{(1)}_q(M) = \dim H^q_{(1)}(M), \quad q \geq n+1.$

Problem: $H^{n}_{(1)}(M)$ can be infinite dimensional

[Fernández-Ibáñez-de León 98] For $q \geq n+1$:

$$b_q - b_{q+2} \leq \ \leq c^{(1)}_q \leq \ \leq b_q + b_{q+1}.$$

- HLC $\Longrightarrow b_q - b_{q+2} = c^{(1)}_q$.
- ω exact $\Longrightarrow b_q + b_{q+1} = c^{(1)}_q$.

k-COEFFECTIVE COHOMOLOGIES I

$\alpha \in \Omega^*(M)$ is k-coeffective if $L^k_\omega(\alpha) := \alpha \wedge \omega^k = 0$. Notation: $\mathfrak{C}^q_{(k)}(M)$.

k-Coeffective cohomology: $H^q_{(k)}(M) = \dfrac{\text{Ker}\,\{d : \mathfrak{C}^q_{(k)}(M) \to \mathfrak{C}^{q+1}_{(k)}(M)\}}{\text{Im}\,\{d : \mathfrak{C}^{q-1}_{(k)}(M) \to \mathfrak{C}^q_{(k)}(M)\}}$.

If (M^{2n}, ω) is of finite type:

$$\underbrace{H^0_{(k)}, \ \ldots, H^{n-k}_{(k)}}_{=0}, \ H^{n-k+1}_{(k)}, \ \underbrace{H^{n-k+2}_{(k)}, \ \ldots, H^{2n-2k+1}_{(k)}, \underbrace{H^{2n-2k+2}_{(k)}, \ \ldots, H^{2n}_{(k)}}_{=H^q}}_{\text{are finite dimensional}}.$$

Problem: $H^{n-k+1}_{(k)}(M)$ can be infinite dimensional

k-COEFFECTIVE COHOMOLOGIES II

Notation: $c_q^{(k)}(M) = \dim H^q_{(k)}(M), \quad q \geq n-k+2.$

$$b_q - b_{q+2k} \leq c_q^{(k)} \leq b_q + b_{q+2k-1}. \tag{1}$$

- HLC $\Longrightarrow b_q - b_{q+2k} = c_q^{(k)}$.
- ω exact $\Longrightarrow b_q + b_{q+2k-1} = c_q^{(k)}$.

Objective

Define a new group for degree $n-k+1$ such that:

- It is finite dimensional.
- Its dimension satisfies similar inequalities to (1)

GENERALIZED COEFFECTIVE COHOMOLOGIES

If (M^{2n},ω) is of finite type, for degree $n-k+1$ we define a new finite-dimensional space (using a long exact sequence in cohomology)

Definition

$$\hat{H}^{n-k+1}(M) = \frac{H^{n-k+1}_{(k)}(M)}{\frac{H^{n+k}(L^k_\omega(\Omega^k(M)))}{H^{n-k}(M)}}. \quad \dim \hat{H}^{n-k+1}(M) = \hat{c}_{n-k+1}.$$

- $b_{n-k+1} - b_{n+k+1} \leq \hat{c}_{n-k+1} \leq b_{n-k+1}$ (HLC and exact $\Rightarrow =$)

Generalized coeffective cohomology

$$\hat{H}^{n-k+1},\ H^{n-k+2}_{(k)}, \ldots, H^{2n}_{(k)}, \quad 1 \leq k \leq n.$$

- $\chi^{(k)}(M) = (-1)^{n-k+1}\hat{c}_{n-k+1} + \sum_{i=n-k+2}^{2n} (-1)^i c_i^{(k)} = \sum^{n+k} (-1)^i b_i$ topological invariant.

Relations between coeffective and harmonic cohomologies

Theorem

Let (M^{2n}, ω) be a symplectic manifold of finite type. The following relation holds for every $k = 1, \ldots, n$:

$$h_{n-k+1} - h_{n+k+1} = \hat{c}_{n-k+1}.$$

No relation for h_{n+1}.

Extensions of the generalized coeffective complexes I

The coeffective complexes are not elliptic in degree $n-k+1$ (that is the reason for which $H^{n-k+1}_{(k)}$ can be infinite dimensional).

Idea: Construct an elliptic extension of the coeffective complexes. ([Eastwood 12] for $k=1$.)

$$\begin{array}{l}
0 \longrightarrow \Omega^0 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{2k-1} \xrightarrow{\check{d}} \check{\Omega}^{2k}_{(k)} \xrightarrow{\check{d}} \cdots \xrightarrow{\check{d}} \check{\Omega}^{n+k-2}_{(k)} \xrightarrow{\check{d}} \check{\Omega}^{n+k-1}_{(k)} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow D \\
0 \longleftarrow \Omega^{2n} \xleftarrow{d} \cdots \xleftarrow{d} \Omega^{2n-2k+1} \xleftarrow{d} \mathfrak{C}^{2n-2k}_{(k)} \xleftarrow{d} \cdots \xleftarrow{d} \mathfrak{C}^{n-k+2}_{(k)} \xleftarrow{d} \mathfrak{C}^{n-k+1}_{(k)}
\end{array}$$

where $\check{\Omega}^q_{(k)}(M) = \dfrac{\Omega^q(M)}{L^k_\omega(\Omega^{q-2k}(M))}$, $\check{d}$ is induced by d, and D is second order operator.

EXTENSIONS OF THE GENERALIZED COEFFECTIVE COMPLEXES II

Cohomology groups: $\check{H}^q_{(k)}(M)$, for $q = 0, \ldots, 2n + 2k - 1$.

$$\underbrace{\check{H}^0_{(k)}, \ \ldots, \check{H}^{2k-2}_{(k)}}_{= H^q}, \ \check{H}^{2k-1}_{(k)}, \ldots, \check{H}^{n+k}_{(k)}, \ \underbrace{\check{H}^{n+k+1}_{(k)}, \ \ldots, \check{H}^{2n+2k-1}_{(k)}}_{= H^{q-2k+1}_{(k)}}.$$

- If (M^{2n}, ω) is of finite type, $\check{c}^{(k)}_q(M) = \dim \check{H}^q_{(k)}(M)$ finite $\forall q$.

$$b_{q-2k+1} - b_{q+1} \leq \check{c}^{(k)}_q \leq b_{q-2k+1} + b_q.$$

  - HLC $\Longrightarrow b_{q-2k+1} - b_{q+1} = \check{c}^{(k)}_q$.
  - ω exact $\Longrightarrow b_{q-2k+1} + b_q = \check{c}^{(k)}_q$.

Relations between coeffective and primitive cohomologies

These last cohomology groups allow us to recover all the primitive cohomology groups [Tseng-Yau 12] via the filtered cohomology groups [Tsai-Tseng-Yau 14]:

$$PH^q_{\partial_+}(M) \cong F^0 H^q_+(M) \cong \check{H}^q_{(1)}(M), \quad 0 \le q \le n-1;$$

$$PH^q_{\partial_-}(M) \cong F^0 H^q_-(M) \cong \check{H}^{2n-q+1}_{(1)}(M) \cong H^{2n-q}_{(1)}(M), \quad 0 \le q \le n-1;$$

$$PH^{n-k+1}_{dd^\Lambda}(M) \cong F^{k-1} H^{n+k-1}_+(M) \cong \check{H}^{n+k-1}_{(k)}(M), \quad 1 \le k \le n;$$

$$PH^{n-k+1}_{d+d^\Lambda}(M) \cong F^{k-1} H^{n+k-1}_-(M) \cong \check{H}^{n+k}_{(k)}(M), \quad 1 \le k \le n.$$

In fact, the following isomorphisms hold:

$$F^{k-1} H^{n+k-s-1}_-(M) \cong H^{n-k+s+1}_{(k)}(M) \cong \check{H}^{n+k+s}_{(k)}(M), \quad 1 \le s \le n+k-1.$$

$$\check{H}^{n+k+s}_{(k)}(M) \cong \check{H}^{n+k-s-1}_{(k)}(M) \text{ for } s = 0, \dots, n+k-1.$$

More relations between cohomologies

Let (M^{2n}, ω) be symplectic of finite type.

$$\boxed{0 \leq \check{c}^{(k)}_{n+k} - \hat{c}_{n-k+1} \leq b_{n+k} - h_{n+k}} \qquad (2)$$

$$\check{c}^{(k)}_{n+k} - \hat{c}_{n-k+1} = b_{n+k} - h_{n+k} \Longleftrightarrow L^k(H^{n-k}_{\mathrm{hr}}(M)) = L^k(H^{n-k}(M)).$$

Particular cases for equality: $k = n,\ n-1,\ n-2 \Longrightarrow$ Explicit description of

$$\check{c}^{(n)}_{2n},\ \check{c}^{(n-1)}_{2n-1},\ \check{c}^{(n-2)}_{2n-2}.$$

Proposition

(M^{2n}, ω) satisfies HLC $\Longrightarrow$ $\check{c}^{(k)}_{n+k} = \hat{c}_{n-k+1}$ for $k = 1, \ldots, n$.

Flexibility

An interesting question in the study of the symplectic harmonicity is the flexibility. It was introduced and studied in [IRTU 01] and [Yan 96] motivated by a question posed by Khesin and McDuff:

Question: On which compact manifolds M does there exist a family ω_t of symplectic forms such that the dimension of $H^q_{\mathrm{hr}}(M, \omega_t)$ varies?

Definition

A manifold M^{2n} is **h**-flexible if M possesses a continuous symplectic family ω_t, $t \in [a, b]$ s.t. $h_q(M, \omega_a) \neq h_q(M, \omega_b)$ for some q.

- [Yan 96] studied the case of closed 4-manifolds:
  - 4-dimensional nilmanifolds (compact quotients of nilpotent Lie groups) are not **h**-flexible.
  - There exist 4-dimensional **h**-flexible manifolds.
- [IRTU 01]: Some 6-dimensional nilmanifolds are **h**-flexible.

COHOMOLOGICAL FLEXIBILITIES

We can define other notions of flexibility:

Definition

A closed smooth manifold M^{2n} is said to be

1. **c**-flexible if M possesses a continuous symplectic family ω_t, $t \in [a, b]$ such that, for some $1 \leq k \leq n$
$$\begin{cases} \hat{c}_{n-k+1}(M, \omega_a) \neq \hat{c}_{n-k+1}(M, \omega_b), & \text{or} \\ c_q^{(k)}(M, \omega_a) \neq c_q^{(k)}(M, \omega_b), & \text{for some } n-k+2 \leq q \leq 2n. \end{cases}$$
2. **f**-flexible if M possesses a continuous symplectic family ω_t, $t \in [a, b]$ such that $\check{c}_q^{(k)}(M, \omega_a) \neq \check{c}_q^{(k)}(M, \omega_b)$ for some $1 \leq k \leq n$ and $0 \leq q \leq 2n + 2k - 1$.

These new notions of flexibility give us a simpler way to provide examples of **h**-flexible manifolds.

Closed 4-dimensional symplectic manifolds

Let (M^4, ω) be a closed symplectic manifold.

Generalized coeffective cohomologies are topological.

Filtered cohomologies ($k = 2$ is topological)

$k = 1$: $\check{c}_0^{(1)}, \check{c}_1^{(1)}, \check{c}_2^{(1)} \,\Big|\, \check{c}_3^{(1)}, \underbrace{\check{c}_4^{(1)}}_{=c_3^{(1)}}, \underbrace{\check{c}_5^{(1)}}_{=c_4^{(1)}}$, where $\boxed{\check{c}_3^{(1)} = b_1 + b_2 - h_3 - 1}$

Harmonic cohomology: $\underbrace{h_0}_{=1}, \underbrace{h_1}_{=b_1}, \underbrace{h_2}_{=b_2}, \underbrace{h_3}_{\text{even}}, \underbrace{h_4}_{=1}$.

- M is never **c**-flexible.
- M is **f**-flexible $\Longleftrightarrow$ M is **h**-flexible. In particular, $\exists\ M^4$ **f**-flexible.
- If $b_1(M) \leq 1$, then M is not **f**-flexible.
- If M is completely solvable solvmanifold, it is not **c**-flexible, **f**-flexible or **h**-flexible.

CLOSED 6-DIMENSIONAL SYMPLECTIC MANIFOLDS

Generalized coeffective cohomologies

$k = 1$:

$$\hat{c}_3,\ c_4^{(1)},\ \underbrace{c_5^{(1)}}_{=b_5},\ \underbrace{c_6^{(1)}}_{=b_6},$$

where

$$\boxed{\hat{c}_3 = c_4^{(1)} + 1 - b_1 - b_2 + b_3}$$

$k = 2$ and $k = 3$ are topological.

Harmonic cohomology

$$\underbrace{h_0}_{=1},\ \underbrace{h_1}_{=b_1},\ \underbrace{h_2}_{=b_2},\ h_3,\ h_4,\ \underbrace{h_5}_{\text{even}},\ \underbrace{h_6}_{=1},$$

where

$$\boxed{\hat{c}_3 = h_3 - h_5}$$

Closed 6-dimensional symplectic manifolds

Filtered cohomologies ($k = 3$ is topological)

$k = 1$:

$$\check{c}_0^{(1)},\ \check{c}_1^{(1)},\ \check{c}_2^{(1)},\ \check{c}_3^{(1)}\ \Big|\ \check{c}_4^{(1)},\ \underbrace{\check{c}_5^{(1)}}_{=c_4^{(1)}},\ \underbrace{\check{c}_6^{(1)}}_{=b_5},\ \underbrace{\check{c}_7^{(1)}}_{=b_6},$$

where $\boxed{\check{c}_4^{(1)} = \hat{c}_3 - h_4 + b_2, \quad \check{c}_5^{(1)} = c_4^{(1)}}$

$k = 2$:

$$\check{c}_0^{(2)},\ \check{c}_1^{(2)},\ \check{c}_2^{(2)},\ \check{c}_3^{(2)},\ \check{c}_4^{(2)}\ \Big|\ \check{c}_5^{(2)},\ \underbrace{\check{c}_6^{(2)}}_{=b_3},\ \underbrace{\check{c}_7^{(2)}}_{=b_4},\ \underbrace{\check{c}_8^{(2)}}_{=b_5},\ \underbrace{\check{c}_9^{(2)}}_{=b_6},$$

where $\boxed{\check{c}_5^{(2)} = b_1 + b_2 - h_5 - 1}$

All the dimensions are determined by Betti numbers and $\hat{c}_3,\ h_4$ and h_5.

FLEXIBILITY IN DIMENSION ≥ 6

Let (M^{2n}, ω) be a closed symplectic manifold.

- Results in dimension 6:

1. M is **c**-flexible $\Longrightarrow$ M is **f**-flexible and **h**-flexible.
2. M is not **c**-flexible $\Longrightarrow$ M is **f**-flexible if and only if it is **h**-flexible.

Examples of **h**-flexible nilmanifolds in dimension 6 can be found in [IRTU 01].
We classify the 6-dimensional **c**-flexible nilmanifolds.

- Results in dimension ≥ 8:

1. M is **c**-flexible $\Longrightarrow$ M is **f**-flexible or **h**-flexible.
2. $\exists$ M **f**-flexible for any dimension $2n$.

Example in dimension 8: a solvmanifold that is **c**-flexible, **f**-flexible and **h**-flexible.

Thank you for your attention