Hypersymplectic structures with torsion on Lie algebroids

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Abstract

We define hypersymplectic structures with torsion on Lie algebroids, address some properties related with compatibilities of tensors and deformations of the Lie algebroid structure. We conclude with an explicit example on $SU(3)$.

1. Hypersymplectic structures with torsion on Lie algebroids

On a Lie algebroid $(A, \rho, [.,.])$,

- take three nondegenerate 2-forms ω_1, ω_2 and $\omega_3 \in \Gamma(\wedge^2 A^*)$,
- consider their inverses π_1, π_2 and $\pi_3 \in \Gamma(\wedge^2 A)$
- and define the *transition morphisms*

$$N_i := \pi_{i-1}^\sharp \circ \omega_{i+1}^\flat, \quad i \in \mathbb{Z}_3.$$

DEFINITION 1.1. *The triplet $(\omega_1, \omega_2, \omega_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure with torsion on A if*

- $N_i{}^2 = \varepsilon_i \mathrm{id}_A, \quad i = 1, 2, 3,$
- $\varepsilon_2 N_1 \mathrm{d}\omega_1 = \varepsilon_3 N_2 \mathrm{d}\omega_2 = \varepsilon_1 N_3 \mathrm{d}\omega_3,$

where the parameters $\varepsilon_i = \pm 1$ form the triple $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and

$$N_i \mathrm{d}\omega_i(X, Y, Z) := \mathrm{d}\omega_i(N_i X, N_i Y, N_i Z),$$

for all $X, Y, Z \in \Gamma(A)$.

If ω_1, ω_2 and ω_3 are closed (i.e., if they are symplectic forms) then $(\omega_1, \omega_2, \omega_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure (without torsion) on A.

PROPOSITION 1.2. *The following assertions are equivalent:*

1. $\varepsilon_2 N_1 \mathrm{d}\omega_1 = \varepsilon_3 N_2 \mathrm{d}\omega_2 = \varepsilon_1 N_3 \mathrm{d}\omega_3$;
2. $\varepsilon_1 [\pi_1, \pi_1]_{SN} = \varepsilon_2 [\pi_2, \pi_2]_{SN} = \varepsilon_3 [\pi_3, \pi_3]_{SN}$, *where $[\cdot,\cdot]_{SN}$ is the Schouten-Nijenhuis bracket on $\Gamma(\wedge^\bullet A)$.*

Proof. Use the fact that when ω is a non degenerate 2-form with inverse bivector π, we have $[\pi, \pi]_{SN}(.,.,.) = 2\mathrm{d}\omega(\pi^\sharp ., \pi^\sharp ., \pi^\sharp .)$ or, equivalently, $\mathrm{d}\omega(.,.,.) = \frac{1}{2}[\pi, \pi]_{SN}(\omega^\flat ., \omega^\flat ., \omega^\flat .)$. □

As a consequence of the previous proposition, we can define an $\boldsymbol{\varepsilon}$-hypersymplectic structure with torsion as follows.

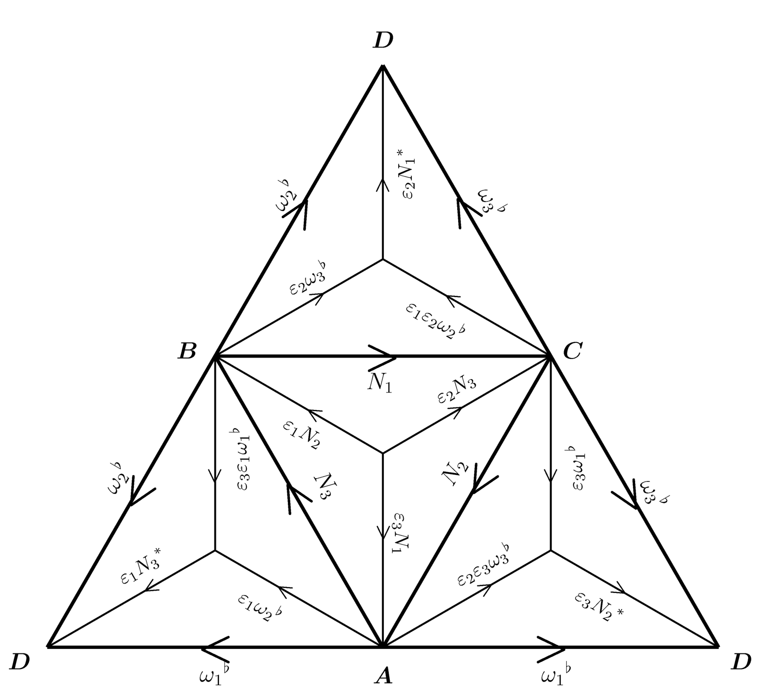
DEFINITION 1.3. *$(\omega_1, \omega_2, \omega_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure with torsion on A if*

- $N_i{}^2 = \varepsilon_i \mathrm{id}_A, \quad i = 1, 2, 3,$
- $\varepsilon_1 [\pi_1, \pi_1]_{SN} = \varepsilon_2 [\pi_2, \pi_2]_{SN} = \varepsilon_3 [\pi_3, \pi_3]_{SN},$

where the parameters $\varepsilon_i = \pm 1$ form the triple $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

▶ We will use the abbreviation $\boldsymbol{\varepsilon}$-HST for $\boldsymbol{\varepsilon}$-hypersymplectic structure with torsion.

▶ The algebraic relations between morphisms ω_i, π_j and N_k in $\boldsymbol{\varepsilon}$-HST structures are summarized in the following diagram:



▶ From now on, we will concentrate our study on the case $\varepsilon_1 \varepsilon_2 \varepsilon_3 = -1$ to have, in particular, anti-commuting morphisms N_i. Then, we have to distinguish two cases:

$\varepsilon_i = -1, \forall i \ \rightarrow$ hypersymplectic with torsion (HST);
otherwise $\rightarrow$ para-hypersymplectic with torsion (para-HST).

2. Properties. Compatibilities. Deformations

One of the important properties of a (para-)HST structure on a Lie algebroid A is the integrability of the transition morphisms $N_i, i = 1, 2, 3$. This was actually an axiom in the original definition [5] of HST structures (more exactly of HKT structures - see Joana's poster for the 1-1 correspondence between HST and HKT). In [3], the authors proved it as a consequence of the remaining axioms. The following theorem generalizes this result for (para-)HST structures.

THEOREM 2.1. *Let $(\omega_1, \omega_2, \omega_3)$ be a (para-)HST structure on a Lie algebroid A. Then, the transition morphisms $N_i, i = 1, 2, 3$, have vanishing Nijenhuis torsion, i.e., they are* integrable *complex or para-complex morphisms.*

Proof. The proof is simple but goes through (para-)HS structures (without torsion) on Courant algebroids (see Joana's poster) and uses some compatibility results there. □

Using the same tools that we use in the previous theorem (i.e., (para-)HS structures on Courant algebroids) we prove relevant compatibilities between tensors induced by a (para-)HST structure.

PROPOSITION 2.2. *Consider $(\omega_1, \omega_2, \omega_3)$ a (para-)HST structure on a Lie algebroid A.*

1. *The inverse bivectors are pairwise compatible, i.e., $[\pi_i, \pi_j]_{SN} = 0, i, j \in 1, 2, 3, i \neq j$.*
2. *The transition morphisms are pairwise compatible, in the sense that $[N_i, N_j]_{FN} = 0, i, j \in 1, 2, 3$, where $[\cdot,\cdot]_{FN}$ is the Frölicher-Nijenhuis bracket.*
3. *The transition morphisms, N_i, and the 2-forms, ω_j, are compatible in the sense that $\mathrm{d}(\mathrm{i}_{N_j}\omega_i) = \mathrm{d}_{N_j}\omega_i, i, j \in 1, 2, 3, i \neq j$.*

▶ If $N : A \rightarrow A$ is a Nijenhuis tensor on a Lie algebroid $(A, \rho, [\cdot,\cdot])$ then we define a new Lie algebroid bracket on $\Gamma(A)$ by setting

$$[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y].$$

THEOREM 2.3. *The triplet $(\omega_1, \omega_2, \omega_3)$ is a (para-)HST on the Lie algebroid $(A, \rho, [\cdot,\cdot])$ if and only if $(\omega_1, \omega_2, \omega_3)$ is a (para-)HST on the Lie algebroid $(A, \rho \circ N_i, [\cdot,\cdot]_{N_i}), i = 1, 2, 3$.*

▶ If $\pi \in \Gamma(\wedge^2 A)$ is a non-degenerate bivector with inverse $\omega \in \Gamma(\wedge^2 A)$, on a Lie algebroid $(A, \rho, [\cdot,\cdot])$, then we define a new Lie algebroid bracket on $\Gamma(A^*)$ by setting

$$[\alpha, \beta]^\pi = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - \mathrm{d}(\pi(\alpha, \beta)) + \frac{1}{2}\omega^\flat([\pi, \pi]_{SN}(\alpha, \beta, \cdot)).$$

Notice that, when π is a Poisson bivector, we recover a well known bracket on $\Gamma(A^*)$.

THEOREM 2.4. *The triplet $(\omega_1, \omega_2, \omega_3)$ is a (para-)HST on the Lie algebroid $(A, \rho, [\cdot,\cdot])$ if and only if (π_1, π_2, π_3) is a (para-)HST on the Lie algebroid$(A^*, \rho \circ \pi_i^\sharp, [\cdot,\cdot]^{\pi_i}), i = 1, 2, 3$.*

Combining the two previous theorems we get the following commutative diagram:

$(\omega_1, \omega_2, \omega_3)$ (para-)HST on $(A, [\cdot,\cdot])$ ⟶ (π_1, π_2, π_3) (para-)HST on $(A^*, [\cdot,\cdot]^{\pi_j})$

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$(\omega_1, \omega_2, \omega_3)$ is (para-)HST on $(A, [\cdot,\cdot]_{N_i})$ ⟶ (π_1, π_2, π_3) (para-)HST on $(A^*, ([\cdot,\cdot]^{\pi_j})_{N_i^*}) = (A^*, ([\cdot,\cdot]_{N_i})^{\pi_j})$

3. Example

To conclude, we address an explicit example of an hypersymplectic structure with torsion on the Lie algebra $\mathfrak{su}(3)$ of the Lie group $SU(3)$.

We write E_{pq} for the elementary 3×3-matrix with 1 at position (p, q) and consider the basis of $\mathfrak{su}(3)$ consisting of eight complex matrices:

$$A_1 = i(E_{11} - E_{22}), \quad A_2 = i(E_{22} - E_{33}),$$
$$B_{pq} = E_{pq} - E_{qp}, \quad C_{pq} = i(E_{pq} + E_{qp}),$$

where $p, q \in \{1, 2, 3\}$ such that $p < q$. We will write $a_1, a_2, \ldots, c_{23}$ for the dual basis

The commutation relations for this basis are collected in the table:

	A_2	B_{12}	C_{12}	B_{13}	C_{13}	B_{23}	C_{23}
A_1	0	$2C_{12}$	$-2B_{12}$	C_{13}	$-B_{13}$	$-C_{23}$	B_{23}
A_2		$-C_{12}$	B_{12}	C_{13}	$-B_{13}$	$2C_{23}$	$-2B_{23}$
B_{12}			$2A_1$	$-B_{23}$	$-C_{23}$	B_{13}	C_{13}
C_{12}				C_{23}	$-B_{23}$	C_{13}	$-B_{13}$
B_{13}					$2(A_1 + A_2)$	$-B_{12}$	C_{12}
C_{13}						$-C_{12}$	$-B_{12}$
B_{23}							$2A_2$

We define a triplet of 2-forms on $SU(3)$:

$$\omega_1 = -\frac{\sqrt{3}}{2}a_1 a_2 + b_{12}c_{12} + b_{13}c_{13} - b_{23}c_{23};$$
$$\omega_2 = \frac{\sqrt{3}}{2}a_2 b_{12} - a_1 c_{12} + \frac{1}{2}a_2 c_{12} - b_{13}b_{23} + c_{13}c_{23};$$
$$\omega_3 = \frac{\sqrt{3}}{2}a_2 c_{12} + a_1 b_{12} - \frac{1}{2}a_2 b_{12} + b_{13}c_{23} + b_{23}c_{13}.$$

These 2-forms are not closed, for example we have

$$\mathrm{d}\omega_1 = -\sqrt{3}a_1(b_{13}c_{13} + b_{23}c_{23}) + \sqrt{3}a_2(b_{12}c_{12} + b_{13}c_{13}) - b_{12}b_{13}c_{23} - b_{12}b_{23}c_{13} - b_{13}b_{23}c_{12} - c_{12}c_{13}c_{23},$$

but they satisfy the HST definition since

$$\mathrm{d}\omega_1(N_1\cdot, N_1\cdot, N_1\cdot) = \mathrm{d}\omega_2(N_2\cdot, N_2\cdot, N_2\cdot) = \mathrm{d}\omega_3(N_3\cdot, N_3\cdot, N_3\cdot)$$
$$= -a_1 b_{13}c_{13} + a_1 b_{23}c_{23} - 2a_1 b_{12}c_{12} - a_2 b_{13}c_{13} - 2a_2 b_{23}c_{23} + a_2 b_{12}c_{12} + b_{23}c_{12}c_{13} + b_{13}c_{12}c_{23} + b_{12}c_{13}c_{23} + b_{12}b_{13}b_{23}.$$

On the bases $(A_1, A_2, B_{12}, B_{13}, B_{23}, C_{12}, C_{13}, C_{23})$ and its dual, matrix representations are given by

$$\omega_1 = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \omega_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\omega_3 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, N_1 = \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, N_3 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 \\ 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathrm{g} := \omega_{i-1}{}^\flat \circ \pi_i{}^\sharp \circ \omega_{i+1}{}^\flat = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

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