XXIII IFWGP – Granada, 2014

AN EXACTLY SOLVABLE DEFORMATION OF THE COULOMB PROBLEM FROM A QUANTUM TAUB–NUT SYSTEM

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Motivation

In this contribution we present the **quantization of the ND Hamiltonian system**

$$\mathcal{H}=\mathcal{T}(\mathbf{q},\mathbf{p})+\mathcal{U}(\mathbf{q})=\frac{|\mathbf{q}|}{2(\eta+|\mathbf{q}|)}\,\mathbf{p}^2-\frac{k}{\eta+|\mathbf{q}|}$$

where η and k are real parameters and $\mathbf{q},\mathbf{p}\in\mathbb{R}^N$ are canonical coordinates and momenta. The system $\mathcal{H}$ can be regarded as an **η-deformation of the ND Euclidean Coulomb problem** with coupling constant k, since its limit $\eta\to 0$ yields

$$\mathcal{H}=\frac{1}{2}\,\mathbf{p}^2-\frac{k}{|\mathbf{q}|}.$$

$\mathcal{H}$ can also be interpreted as a **Hamiltonian defined on a curved manifold $\mathcal{M}$** with **metric**

$$\mathrm{d}s^2=\left(1+\frac{\eta}{|\mathbf{q}|}\right)\mathrm{d}\mathbf{q}^2$$

and whose **scalar curvature** is given by

$$R=\eta(N-1)\,\frac{4(N-3)r+3\eta(N-2)}{4r(\eta+r)^3},\qquad r=|\mathbf{q}|.$$

The domain of r in $\mathcal{M}$ depends on η: if $\eta>0, r\in(0,\infty)$ and if $\eta<0, r\in(|\eta|,\infty)$. Thus, we have a **two-parametric family (η,k) of ND systems** in which:

- If $\eta>0$, we have a system on a space with **positive scalar curvature if $N\geq 3$**.
- If $\eta>0$ and $N=2$ the space has **negative** scalar curvature and is **asymptotically flat**.
- When $N=3$, the case with $\eta>0$ and $k<0$ describes the **3D reduction of the geodesic motion on the 4D Eucliden Taub-NUT metric.** [1]

1. Maximal superintegrability of the classical system

The Hamiltonian $\mathcal{H}$ defines a **maximally superintegrable classical system** since it is endowed with $(2N-1)$ functionally independent integrals of motion. Explicitly:[2]

Proposition. **(i)** The Hamiltonian $\mathcal{H}$ is endowed with $(2N-3)$ **angular momentum integrals** given by $(m=2,\ldots,N)$

$$C^{(m)}=\sum_{1\leq i<j\leq m}(q_ip_j-q_jp_i)^2,\quad C_{(m)}=\sum_{N-m<i<j\leq N}(q_ip_j-q_jp_i)^2,\quad C^{(N)}=C_{(N)}\equiv\mathbf{L}^2,$$

where $\mathbf{L}^2$ is the square of the **total angular momentum**.

(ii) The Hamiltonian $\mathcal{H}$ Poisson-commutes with the $\mathcal{R}_i$ components $(i=1,\ldots,N)$ of the **Runge–Lenz N-vector** given by

$$\mathcal{R}_i=\sum_{j=1}^{N}p_j(q_jp_i-q_ip_j)+\frac{q_i}{|\mathbf{q}|}\,(\eta\mathcal{H}+k)\,.$$

(iii) The set $\{\mathcal{H},C^{(m)},C_{(m)},\mathcal{R}_i\}$, with $m=2,\ldots,N$ and a fixed index i, is formed by $(2N-1)$ **functionally independent functions**.

Note that the Runge–Lenz vector $\mathbf{R}$, the angular momentum $\mathbf{L}$ and the Hamiltonian $\mathcal{H}$ are related in the form: $\mathbf{R}^2=\sum_{i=1}^{N}\mathcal{R}_i^2=2\mathbf{L}^2\mathcal{H}+(\eta\mathcal{H}+k)^2$.

$\mathcal{H}$ can also be expressed in terms of **hyperspherical coordinates** (r,θ_j). Thus, for a given value of $\mathbf{L}^2$, the Hamiltoinian $\mathcal{H}$ can be written as a **1D radial system**:

$$\mathcal{H}(r,p_r)=\mathcal{T}(r,p_r)+\mathcal{U}(r)=\frac{r}{2(\eta+r)}\left(p_r^2+\frac{\mathbf{L}^2}{r^2}\right)-\frac{k}{\eta+r}.$$

2. Maximally superintegrable quantization

We shall make use of the **conformal Laplacian quantization** [3]

$$\hat{\mathcal{H}}_{\rm c}=-\frac{\hbar^2}{2}\Delta_{\rm c}+\mathcal{U}=-\frac{\hbar^2}{2}\left(\Delta_{\rm LB}-\frac{(N-2)}{4(N-1)}\,R\right)+\mathcal{U}\qquad\text{where}$$

- $\Delta_{\rm c}$ is the **conformal Laplacian.**[4]
- $\Delta_{\rm LB}$ is the usual **Laplace–Beltrami operator** on the curved manifold $\mathcal{M}$;
- R is the **scalar curvature** (this term vanishes for $N=2$).

Furthermore, $\hat{\mathcal{H}}_{\rm c}$ can be related through a **similarity transformation** to the Hamiltonian obtained by means of the so-called **direct Schrödinger quantization**

$$\hat{\mathcal{H}}=\frac{-\hbar^2|\mathbf{q}|}{2(\eta+|\mathbf{q}|)}\,\Delta-\frac{k}{\eta+|\mathbf{q}|}$$

(Δ is the Laplacian in the $\mathbf{q}$ coordinates). In this way, the following results can be proven:[5] [6]

[1] T. Iwai, N. Katayama, J.. Phys. A: Math. Gen. 27 (1994) 3179.
[2] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, D. Riglioni, SIGMA 7 (2011) 048.
[3] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, D. Riglioni, Ann. Phys. 326 (2011) 2053.
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[5] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, D. Riglioni, J. Phys: Conf. Series 474 (2013) 012008.
[6] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, D. Riglioni, arXiv:1407.1401.

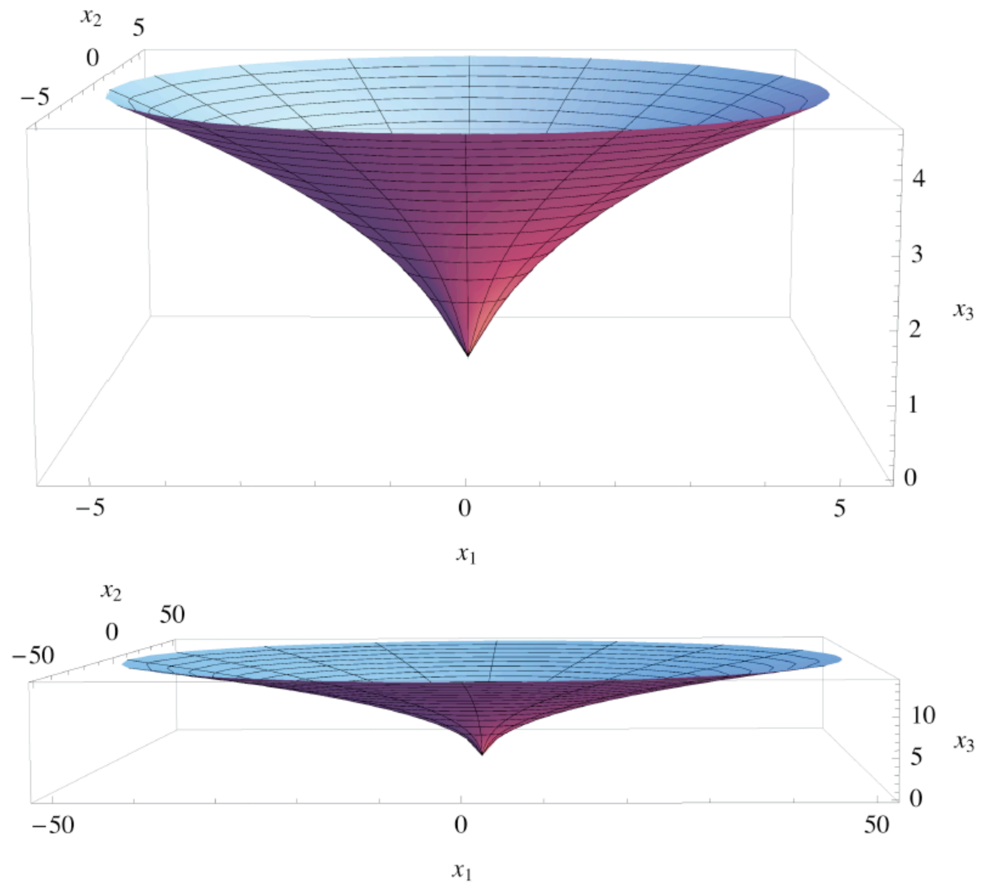

Figure 1: 3D Euclidean embedding of the $N=2$ Taub-NUT space with $\eta=1$, where $r\in[0,5]$ and $r\in[0,50]$.

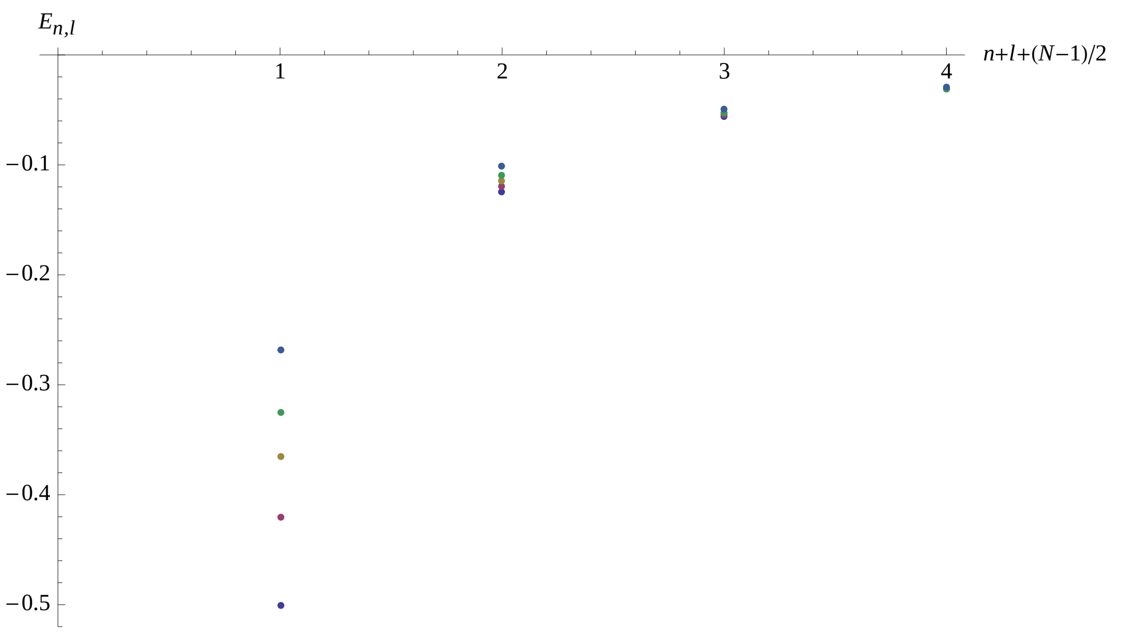

Figure 2: Discrete spectrum for the fundamental and the three first excited states of the Hamiltonian $\hat{\mathcal{H}}_{\rm c}$ when $\eta=\{0,0.2,0.4,0.6,1\}$ with $\hbar=k=1$ and $N\geq 3$. Note that the effect of the η deformation is quite strong for the fundamental state, since it comes from the shift $r\to r+\eta$ in the usual Coulomb potential.

Proposition. **(i)** The quantum Hamiltonian $\hat{\mathcal{H}}_{\rm c}$ commutes with the $(2N-3)$ **quantum angular momentum operators** $\hat{C}^{(m)}=\sum_{1\leq i<j\leq m}(\hat{q}_i\hat{p}_j-\hat{q}_j\hat{p}_i)^2$, $\hat{C}_{(m)}=\sum_{N-m<i<j\leq N}(\hat{q}_i\hat{p}_j-\hat{q}_j\hat{p}_i)^2$, $\hat{C}^{(N)}=\hat{C}_{(N)}=\hat{\mathbf{L}}^2$, as well as with the following N **operators of Runge–Lenz type** $(i=1,\ldots,N)$:

$$\hat{\mathcal{R}}_{{\rm c},i}=\frac{1}{2}\sum_{j=1}^{N}\left(\hat{p}_j+\mathrm{i}\hbar\,\eta\,\frac{(N-2)\hat{q}_j}{4(\eta+|\hat{\mathbf{q}}|)\,\hat{\mathbf{q}}^2}\right)(\hat{q}_j\hat{p}_i-\hat{q}_i\hat{p}_j)$$
$$+\frac{1}{2}\sum_{j=1}^{N}(\hat{q}_j\hat{p}_i-\hat{q}_i\hat{p}_j)\left(\hat{p}_j+\mathrm{i}\hbar\,\eta\,\frac{(N-2)\hat{q}_j}{4(\eta+|\hat{\mathbf{q}}|)\,\hat{\mathbf{q}}^2}\right)+\frac{\hat{q}_i}{|\hat{\mathbf{q}}|}\left(\eta\hat{\mathcal{H}}_{\rm c}+k\right).$$

(ii) Each of the three sets $\{\hat{\mathcal{H}}_{\rm c},\hat{C}^{(m)}\}$, $\{\hat{\mathcal{H}}_{\rm c},\hat{C}_{(m)}\}$ $(m=2,\ldots,N)$ and $\{\hat{\mathcal{R}}_{{\rm c},i}\}$ $(i=1,\ldots,N)$ is formed by N algebraically independent commuting operators.

(iii) The set $\{\hat{\mathcal{H}}_{\rm c},\hat{C}^{(m)},\hat{C}_{(m)},\hat{\mathcal{R}}_{{\rm c},i}\}$ for $m=2,\ldots,N$ with a fixed index i is formed by $(2N-1)$ algebraically independent operators.

(iv) $\hat{\mathcal{H}}_{\rm c}$ is formally self-adjoint on the Hilbert space $L^2(\mathcal{M})$ with its natural scalar product

$$\langle\Psi|\Phi\rangle_{\rm c}=\int_{\mathbb{R}^N}\overline{\Psi(\mathbf{q})}\,\Phi(\mathbf{q})\,\left(1+\frac{\eta}{|\mathbf{q}|}\right)^{N/2}\mathrm{d}\mathbf{q}.$$

Theorem. Let $\hat{\mathcal{H}}_{\rm c}$ be the quantum Hamiltonian with $k>0$ and $\eta>0$. Then:

(i) The **continuous spectrum** of $\hat{\mathcal{H}}_{\rm c}$ is given by $[0,\infty)$. Moreover, there are no embedded eigenvalues and the singular spectrum is empty.

(ii) $\hat{\mathcal{H}}_{\rm c}$ has an infinite number of eigenvalues $E_{n,l}$, depending only on the sum $(n+l)$ and accumulating at 0.

(iii) The **eigenvalues** $E_{n,l}$ of $\hat{\mathcal{H}}_{\rm c}$ are of the form

$$E_{n,l}=\frac{-k^2}{\hbar^2\left(n+l+\frac{N-1}{2}\right)^2+k\eta+\sqrt{\hbar^4\left(n+l+\frac{N-1}{2}\right)^4+2\,\hbar^2k\eta\left(n+l+\frac{N-1}{2}\right)^2}},$$

and $\Psi_{\rm c}=\Phi_{\rm c}(r)Y(\boldsymbol{\theta})$ is an **eigenfunction** of $\hat{\mathcal{H}}_{\rm c}$ with eigenvalue $E_{n,l}$, where

$$\Phi_{\rm c}(r)=\left(1+\frac{\eta}{r}\right)^{\frac{2-N}{4}}r^l\exp\left(-\frac{Kr}{\hbar^2\left(n+l+\frac{N-1}{2}\right)}\right)L_n^{2l+N-2}\left(\frac{2Kr}{\hbar^2\left(n+l+\frac{N-1}{2}\right)}\right),$$

where K **depends on η and $E_{n,l}$**, and L are generalized Laguerre polynomials.

Remarks:

- The bound states of this system satisfy

$$\lim_{n,l\to\infty}E_{n,l}=0,\qquad\lim_{\mathfrak{n}\to\infty}(E_{\mathfrak{n}+1}-E_{\mathfrak{n}})=0,\qquad\mathfrak{n}=n+l.$$

- The limit $\eta\to 0$ of $E_{n,l}$ provides the well-known formula for the standard Coulomb eigenvalues $E^0_{n,l}$. In fact, we have the **perturbative series**

$$E_{n,l}=E^0_{n,l}+\eta\,\frac{k^3}{2\hbar^4\left(n+l+\frac{N-1}{2}\right)^4}-\eta^2\,\frac{5k^4}{8\hbar^6\left(n+l+\frac{N-1}{2}\right)^6}+O(\eta^3).$$

Acknowledgments

This work was partially supported by the Spanish MINECO through the Ramón y Cajal program (A.E.) and under grants MTM2010-18556 (A.B and F.J.H.), FIS2011-22566 (A.E.), by the ICMAT Severo Ochoa under grant SEV-2011-0087 (A.E.), by Banco Santander-UCM under grant GR35/10-A-910556 (A.E.), and by a postdoctoral fellowship by the Laboratory of Mathematical Physics of the CRM, Université de Montréal (D.R.).