Geodesics in static cylindrically symmetric vacuum spacetimes with a cosmological constant

Irene Brito, M. F. A. Da Silva, Filipe C. Mena, N. O. Santos

ireneb@math.uminho.pt, mfasnic@gmail.com, fmena@math.uminho.pt, n.o.santos@qmul.ac.uk

Universidade do Minho
Escola de Ciências

Centro de Matemática

Abstract

We consider static cylindrically symmetric vacuum spacetimes, the Levi-Civita spacetime and the Linet-Tian spacetime [1][2], which includes a cosmological constant. We analyse the geodesics of the Linet-Tian spacetime and compare their dynamics with those of the geodesics in the Levi-Civita spacetime. In particular, we study the effects that the introduction of a cosmological constant has on the orbits' stability [3][4].

Introduction

The Linet-Tian (LT) metric generalizes the static cylindrically symmetric Levi-Civita (LC) metric with non zero Λ and is given by [1, 2]

$$ds^2 = -f dt^2 + d\rho^2 + g dz^2 + l d\phi^2,$$

where $\Sigma = 1 - 2\sigma + 4\sigma^2$, $f = a^2 Q^{2/3} P^{-2(1-8\sigma+4\sigma^2)/3\Sigma}$, $g = b^2 Q^{2/3} P^{-2(1+4\sigma-8\sigma^2)/3\Sigma}$, $l = c^2 Q^{2/3} P^{4(1-2\sigma-2\sigma^2)/3\Sigma}$ and for $\Lambda > 0$,

$$P = \frac{2}{\sqrt{3\Lambda}} \tan R, \quad Q = \frac{1}{\sqrt{3\Lambda}} \sin(2R), \quad R = \frac{\sqrt{3\Lambda}}{2}\rho,$$

b, c and $\sigma \geq 0$ being constants and $\rho \in (0, \pi/\sqrt{3\Lambda})$. In the limit $\Lambda \to 0$, the metric reduces to the Levi-Civita metric [5], in which case $P = Q = \rho$. In the case $\Lambda < 0$, the trigonometric functions are replaced by hyperbolic functions and the LT metric has only one curvature singularity at $\rho = 0$.
The geodesics equations for LT imply

$$\dot t = \frac{E}{f}, \quad \dot\rho^2 = \frac{E^2}{f} - \epsilon - \frac{P_z^2}{g} - \frac{L_z^2}{l}, \quad \dot z = \frac{P_z}{g}, \quad \dot\phi = \frac{L_z}{l},$$

where the dot stands for differentiation with respect to an affine parameter, $\epsilon = 0$, 1 or -1 if the geodesics are, respectively, null, timelike or spacelike, and the constants E, P_z and L_z represent, respectively, the total energy of the test particle, its momentum along the z axis and its angular momentum about the z axis.
In order to study geodesics along ρ, we derive

$$\dot\rho^2 = Q^{-2/3} P^{2(1-8\sigma+4\sigma^2)/3\Sigma}[E^2 - V(\rho)],$$

where the potential

$$V(\rho) = \epsilon Q^{2/3} P^{-2(1-8\sigma+4\sigma^2)/3\Sigma} + (P_z/b)^2 P^{8\sigma(1-\sigma)/\Sigma} + (L_z/c)^2 P^{-2(1-4\sigma)/\Sigma}$$

is positive for null or timelike geodesics.

Planar geodesics ($\dot z = 0$): $\epsilon = 0$ and $\sigma > 1/4$

For the LC spacetime, there is always geodesic confinement if $L_z \neq 0$.
For $\Lambda < 0$, if $E^2 \leq V_\infty$, where $V \to V_\infty = \left(2/\sqrt{3|\Lambda|}\right)^{-2(1-4\sigma)/\Sigma} (L_z/c)^2$, when $\rho \to \infty$, and $L_z \neq 0$, incoming null particles have increasing negative acceleration and hit the axis with infinite speed. However, outgoing null particles move with decreasing negative acceleration and decreasing speed, attaining a maximum distance from the axis $\rho_{max} = \frac{2}{\sqrt{3|\Lambda|}} \tanh\left(\frac{\sqrt{3|\Lambda|}}{2}\rho_{LCmax}\right)$, where ρ_{LCmax} is the maximum value of ρ in the LC spacetime, showing that $|\Lambda|$ increases the maximum distance to the axis z reached by the null particle, see Figure 1. If $E^2 > V_\infty$, the orbit confinement of null geodesics in LC is unstable with the introduction of $\Lambda < 0$.
For $\Lambda > 0$, if $E^2 > V$, where $V \to 0$, when $\rho \to 0$, and $V \to +\infty$, when $\rho \to \pi/\sqrt{3\Lambda}$, incoming particles approach the axis with infinite speed and negative acceleration. Outgoing particles move with decreasing speed and negative acceleration, attaining a maximum distance from the axis $\rho_{max} = \frac{2}{\sqrt{3|\Lambda|}} \tan\left(\frac{\sqrt{3|\Lambda|}}{2}\rho_{LCmax}\right)$, which shows that Λ decreases the maximum distance to the axis reached by the null particle, see Figure 1.

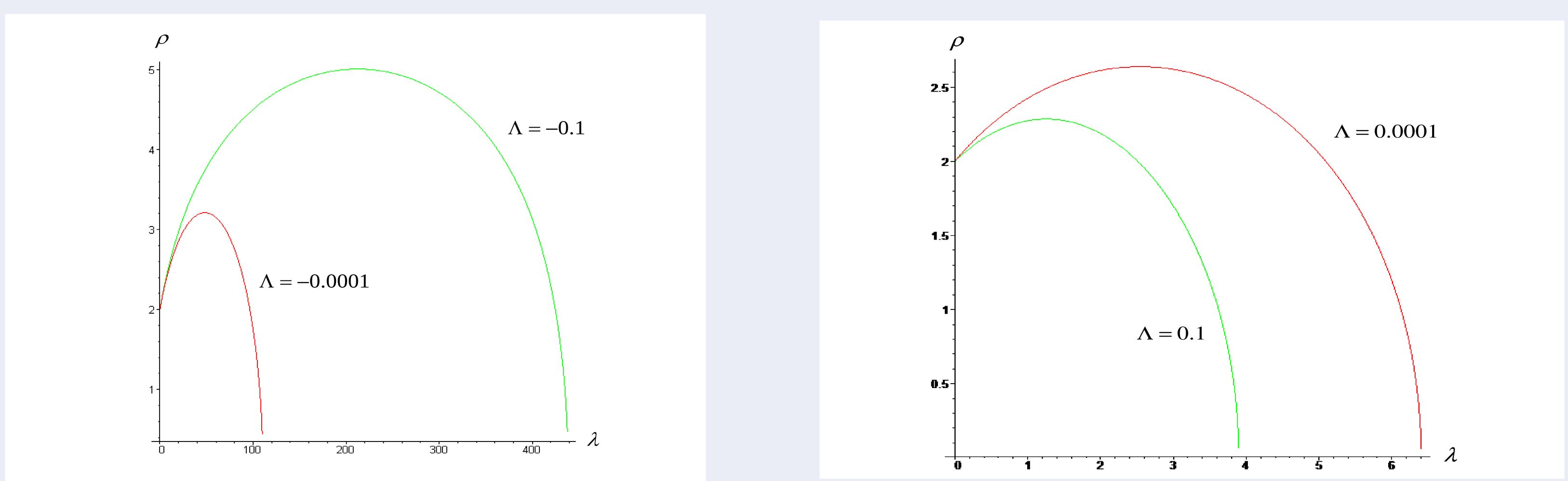

Figure 1: Graphs of the numerical integration of the geodesics' equations along $\rho(\lambda)$ for $\Lambda < 0$, $E = 0.15$, $L_z = 0.1$, $c = 1$ and $\sigma = 0.4$ (on the left) and $\Lambda > 0$, $E = 2$, $L_z = c = 1$ and $\sigma = 0.4$ (on the right).

Non-planar geodesics: $\epsilon = 0$ and $\sigma < 1/4$

In the LC spacetime, there is always geodesic confinement in the radial direction, while in the LT spacetime with $\Lambda < 0$ the geodesics can escape to infinity, the geodesic motion in the LC spacetime being thus unstable with respect to the introduction of $\Lambda < 0$. For $\Lambda < 0$, if $E^2 > V_\infty$, a null particle approaches the axis with decreasing negative acceleration and increasing speed, until it arrives at its minimum distance from the axis at $E^2 = V(\rho)$, where it has vanishing speed. From there on, the null particle is reflected escaping to infinity, see Figure 2.

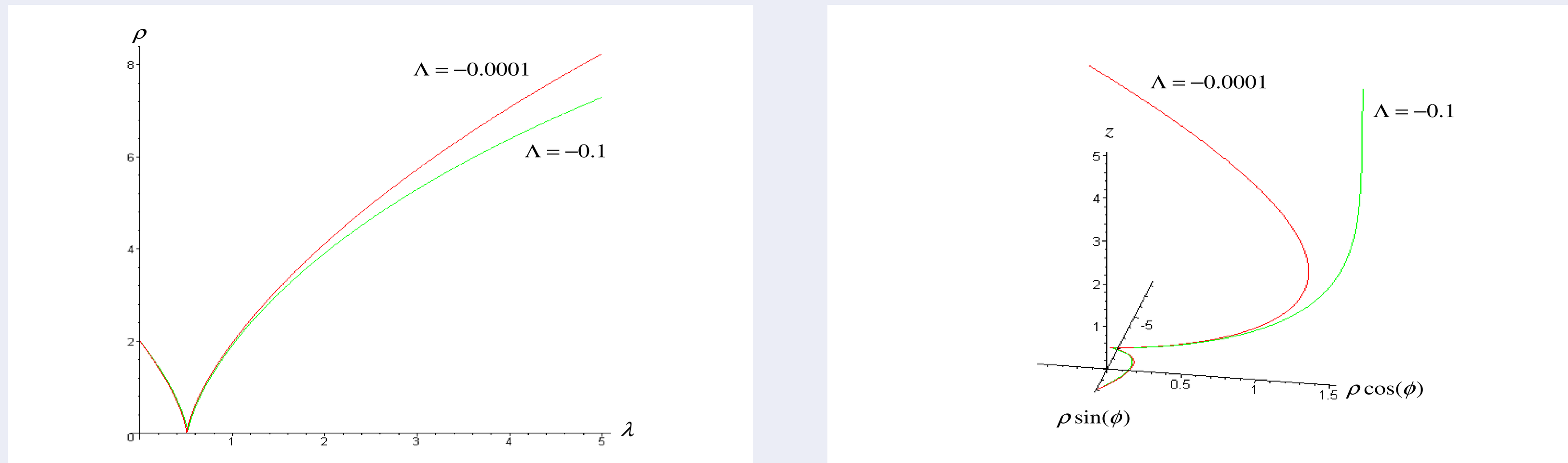

Figure 2: Graphs of the numerical integration of the geodesics' equations along $\rho(\lambda)$ for $\Lambda < 0$, $E = 4$, $L_z = P_z = c = 1$ and $\sigma = 0.2$.

For $\Lambda > 0$, incoming particles approaching the axis z are reflected at $\rho = \rho_{min}$, where they attain $\dot\rho = 0$, and move outwards until they attain again $\dot\rho = 0$ at $\rho = \rho_{max}$ where they are reflected backwards. This trajectory is repeated endlessly, see Figure 3.

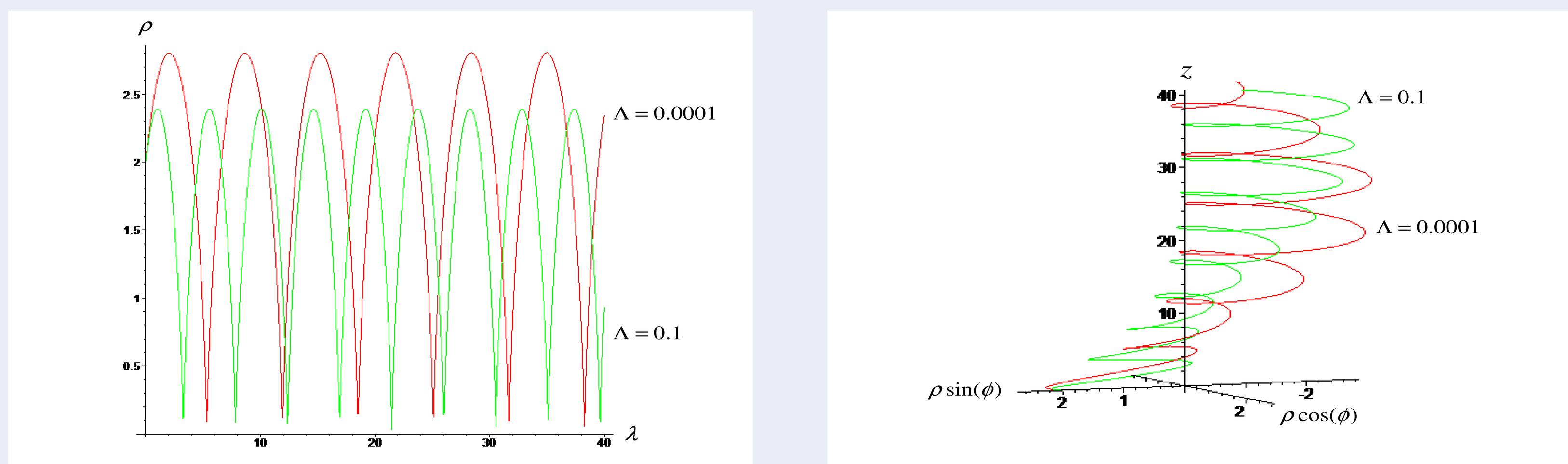

Figure 3: Graphs of the numerical integration of the geodesics' equations along $\rho(\lambda)$ for $\Lambda > 0$, $E = 2$, $L_z = P_z = 0.8$, $c = 1$ and $\sigma = 0.2$.

Non-planar geodesics: $\epsilon \neq 0$

For $\Lambda < 0$ there is always geodesic confinement in the radial direction of the particle, like in the LC spacetime [6]. Examples are plotted in Figure 4, where increasing $|\Lambda|$ decreases the extreme values of the geodesics' distance to the axis.

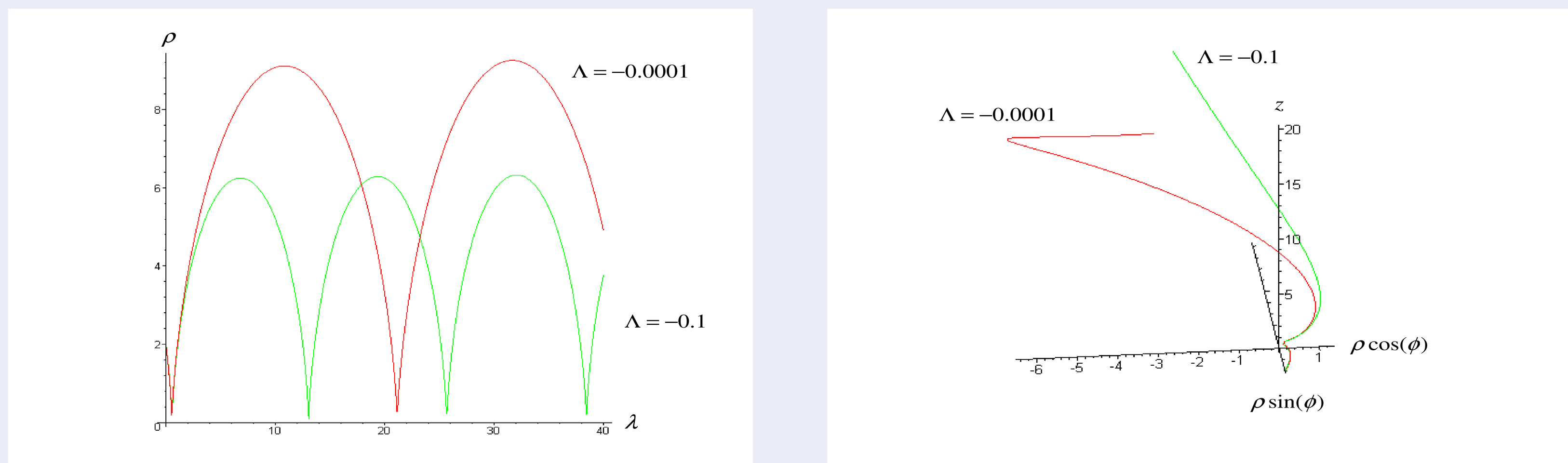

Figure 4: Graphs of the numerical integration of the non-planar geodesics' equations along $\rho(\lambda)$ for $\Lambda < 0$, $E = 4$, $L_z = P_z = c = \epsilon = 1$ and $\sigma = 0.2$.

For $\Lambda > 0$, the geodesics are only confined if $\sigma < 1/4$. If $\sigma \geq 1/4$, incoming particles hit the axis and outgoing particles reach a maximum distance before turning back to the axis, see Figure 5.

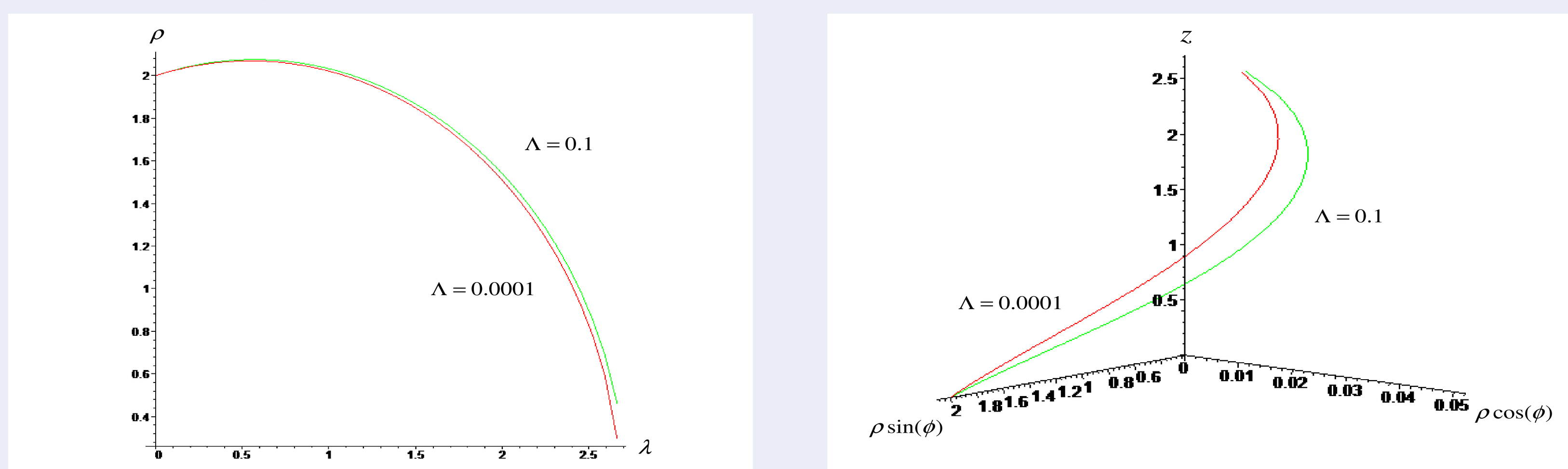

Figure 5: Graphs of the numerical integration of the non-planar geodesics' equations along $\rho(\lambda)$ for $\Lambda > 0$, $E = 2$, $L_z = 0.25$, $P_z = 0.05$, $c = \epsilon = 1$ and $\sigma = 0.4$.

References

[1] B. Linet: *The static, cylindrically symmetric strings in general relativity with cosmological constant.* J. Math. Phys. **27**, 1817 (1986).

[2] Q. Tian: *Cosmic strings with cosmological constant.* Phys. Rev. D **33**, 3549 (1986).

[3] Irene Brito, M. F. da Silva, Filipe C. Mena, N. O. Santos: *Geodesics dynamics in the Linet-Tian spacetime with* $\Lambda < 0$. Gen. Rel. Grav. **46**, 1681 (23 pages) (2014).

[4] Irene Brito, M. F. da Silva, Filipe C. Mena, N. O. Santos: *Geodesics in cylindrically symmetric spacetimes with a positive cosmological constant.* submitted (2014).

[5] T. Levi-Civita: *Rend. Acc. Lincei* **28**, 101 (1919).

[6] L. Herrera, N. O. Santos: *Geodesics in Lewis space-time* J. Math. Phys. **39**, 3817 (1998).