Dual characterizations of the sphere and the hyperbolic plane

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Abstract. By means of a purely synthetic technique, we get a new characterization of the sphere without any curvature conditions, nor completeness or compactness. As well as a dual result for the hyperbolic plane, the spacelike sphere in the Minkowski space.

Some classical characterizations of the sphere involving the curvature

In 1897 Hadamard proved

> Any compact connected regular surface with positive Gaussian curvature in the three-dimensional Euclidean space $\mathbb{E}^3$ is a topological sphere, [3].

His result motivated the search for conditions to conclude that such a surface is necessarily an Euclidean sphere. Two answers were given by Liebmann. The first one proved the rigidity of the sphere, conjectured by F. Minding in 1939.

> If S is a compact and connected regular surface in $\mathbb{E}^3$ with constant Gaussian curvature K, then M is a sphere of radius $1/\sqrt{K}$, [5].
>
> Any compact and connected regular surface in $\mathbb{E}^3$ with positive Gaussian curvature and constant mean curvature is a sphere, [6].

Another well-known characterization of the sphere involving the mean curvature is the Alexandrov theorem:

> A compact and connected regular surface of constant mean curvature in $\mathbb{E}^3$ is a sphere, [1].

Dual results on the sphere and the hyperbolic plane involving the curvature

In the Minkowski space $\mathbb{L}^3$, the hyperbolic plane $\mathbb{H}^2$ can be realized as one connected component of the hyperboloid of two sheets, and so it can be viewed as the spacelike (its induced metric is Riemannian) sphere in $\mathbb{L}^3$.

Hopf get the following dual characterizations of the spheres in $\mathbb{E}^3$ and $\mathbb{L}^3$, [4].

> The sphere is the only complete and simply connected regular surface in $\mathbb{E}^3$ with positive constant Gaussian curvature.
>
> The hyperbolic plane is the only spacelike regular surface in $\mathbb{L}^3$ complete, simply connected and with negative constant Gaussian curvature.

In our paper we are interested in surfaces foliated by circles. In this direction, R. López proved the following dual results

> A regular surface in $\mathbb{E}^3$ with constant Gaussian curvature and foliated by pieces of circles is included in a sphere, or the planes containing the circles of the foliation are parallel, [7].
>
> A regular spacelike surface in $\mathbb{L}^3$ with constant Gaussian curvature and foliated by pieces of circles must be a portion of a hyperbolic plane, unless the planes of the foliation are parallel, [8].

Our results - The sphere

> Let S be a connected regular surface in the Euclidean space $\mathbb{E}^3$ such that its intersection with any affine plane is empty or a circle (including the case with radius zero), then S is necessarily an Euclidean sphere.

The proof

Let $Q \in S$ be an arbitrary point. We consider the tangent plane $T_Q S$ and its normal line through Q, $\mathcal{L}$ (fig.1). We take the sheaf of affine planes with axis $\mathcal{L}$ and we denote by $\{C_i\}_{i\in I}$ the family of circles obtained when intersecting them with S.

We consider a plane Π_0 parallel to $T_Q S$ such that $C = \Pi_0 \cap S$ is a non degenerate circle and we denote $P = \Pi_0 \cap \mathcal{L}$ (fig.2). Then $\Pi_0 \cap C_i \neq \emptyset$ for all $i \in I$, and the intersection points of each circle C_i with Π_0 are the opposite points of a chord of C_i contained in Π_0 with midpoint P. Therefore, the point P must be the center of C and as a direct consequence the circles C_i have all the same radius. Thus, the sphere given by $\bigcup_{i\in I} C_i$ is contained in S. We finish the proof thanks to the connectedness of S.

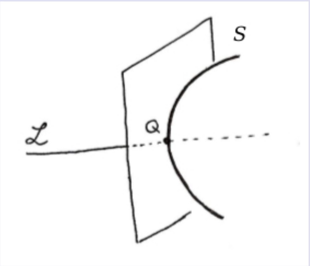

Figure 1

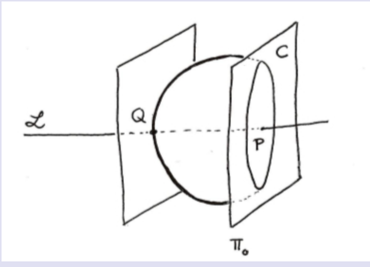

Figure 2

Our results - The hyperbolic plane

> Let S be a spacelike connected regular surface in the Lorentz-Minkowski space $\mathbb{L}^3$ such that its intersection with any spacelike affine plane is empty or a circle (including the case with radius zero), then S is necessarily a hyperbolic plane.

Remark 1 *Notice that a circle in a spacelike affine plane Π of $\mathbb{L}^3$ is the locus of the points in Π at a constant distance from a fixed point in Π, where the distance considered is the one associated to the induced metric.*

The proof (sketch)

Let Π_0 be a spacelike plane such that $C = \Pi_0 \cap S$ is a non-degenerate circle. We denote its center by P and the normal line through P by $\mathcal{L}$ (fig. 3).

Firstly, we prove that $\mathcal{L} \cap S \neq \emptyset$. We notice that S must by closed, and so we can take a point Q at which the distance from P to $\mathcal{L} \cap S$ is attained.

For each $A \in C$ and for each chord perpendicular to the segment AP, we call its midpoint A_m (fig.4). We can chose the chord as close to A as necessary so that the plane generated by it and the segment $A_m Q$ is spacelike, we denote it by Π_{A_m}. We define ε_A to be the supremum (in the set of all possible chords satisfying the previous property) of the distance from A to A_m.

If $\varepsilon = \min_C \varepsilon_A$, we take $0 < \rho < \varepsilon$ and for each $A \in C$ we consider the chord with $d(A_m, A) = \rho$. Hence, all the circles $\Pi_{A_m} \cap S$ (fig.4) have the same radius, and so there is a hyperbolic cap contained in S and containing C.

Finally, for each point $A \in S$ there exists a spacelike plane intersecting S in a non-degenerate circle containing A. Therefore, there exists a hyperbolic cap contained in S and containing A. We finish the proof by using a connectedness argument.

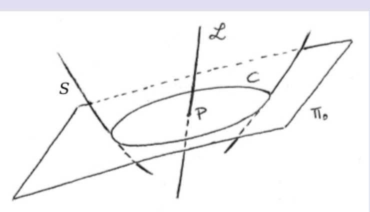

Figure 3

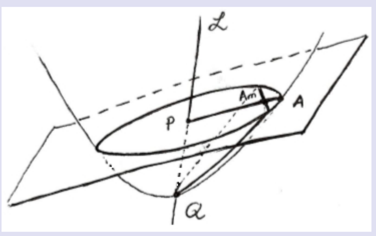

Figure 4

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