Lightlike sets with applications to the rigidity of null geodesic incompleteness

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In this work, we prove rigidity results for generalized plane waves and certain globally hyperbolic spacetimes in the presence of maximal compact surfaces. Motivated by some general properties appearing in these proofs, we develop the theory of *lightlike sets*, entities similar to achronal sets, but more appropriate to deal with low-regularity null submanifolds.

I. INTRODUCTION

In the classic Penrose singularity theorem, one assumes the existence of a closed trapped surface Σ in spacetime. The condition that Σ be a closed trapped surface was Penrose's geometric surrogate of "critical matter density" having been reached in its neighbourhood, signalling an impending gravitational collapse. Mathematically, this condition means (under suitable orientability assumptions) that the so-called *null mean curvatures (or expansion scalars)* $\theta_\pm : \Sigma \to \mathbb{R}$, which measure the initial divergence of the two families of normal null geodesics emanating from Σ, are both strictly negative, i.e., $\theta_\pm < 0$. With additional physically natural assumptions, the null geodesic incompleteness of spacetime then follows.

Natural as this scheme may seem, however, rigorous proofs of the existence of trapped surfaces in the presence of large concentrations of matter can be very difficult to come by in spacetimes without special symmetries. Much more natural entities whose existence has been proven in a number of physically motivated contexts are marginally outer trapped surfaces (MOTS), for which one has $\theta_+ \equiv 0$. MOTS are naturally related to closed trapped surfaces, since they often appear as boundaries of compact spatial regions in spacetime containing closed trapped surfaces and as spatial sections of stationary black hole horizons.

A natural question is then: how does geodesic incompleteness fare when one requires that the spacetime contains a MOTS instead of a closed trapped surface, say in Penrose's singularity theorem? Of course, simple examples obtained by making identifications in Minkowski spacetime show that one need not have geodesic incompleteness in this case. However, it has recently been shown [1, 2, 4] that under reasonable additional assumptions, spacetimes containing MOTS are *generically* geodesically incomplete (in a precise sense - see especially [1]). In other words, geodesic incompleteness fails only in "special situations", being therefore rigid.

In this paper, we prove rigidity theorems involving *maximal* submanifolds, i.e., those having zero mean curvature. These are closely related to MOTS, and also appear in various natural situations. Indeed, the model for our basic rigidity result (cf. Theorem IV.1 below) is Theorem 7.1 of Ref. [4], which is a rigidity result for MOTS, and our proof therein is an adaptation of the proof in [4]. Theorem IV.1 will in turn serve as a basis to obtain a few other rigidity results, which provide descriptions of certain classes of geodesically complete spacetimes of physical interest. In addition, most of our results remain valid for any spacetime dimension larger than two without assuming any field equations.

The general properties of null geodesics and achronal sets appearing in the proofs have led us to investigate certain subsets of spacetime of independent interest, which we have christened

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lightlike sets. These sets generalize a concept first introduced by G. Galloway in [5] (cf. Definition 3.1 of that reference).

II. PRELIMINARIES

In all that follows, we fix a *spacetime*, i.e, an $(n+1)$-dimensional ($n \geq 2$), second-countable, connected, Hausdorff, smooth (i.e., C^∞) Lorentz manifold M endowed with a smooth metric tensor g (signature $(-,+,\ldots,+)$), and with a fixed time-orientation. All submanifolds of M are regarded as C^∞, embedded, and their topology is the induced topology. Finally, we follow the convention that nonspacelike vectors are always nonzero, and we use the terms "nonspacelike" and "causal" interchangeably. If there is no risk of confusion, we shall often use the notation $\langle\,.\,,.\rangle = g(\,.\,,.)$.

By a *surface* we will always mean a connected, acausal (hence spacelike), codimension 2 submanifold $\Sigma \subset M$. We say that the surface Σ is *two-sided* if its normal bundle $N\Sigma$ is trivial. In this case, we can pick two linearly independent, normal future-directed null vector fields $K_\pm : \Sigma \to N\Sigma$ and define the *null mean curvatures* $\theta_\pm \in C^\infty(\Sigma)$ of Σ by

$$\theta_\pm(p) = -\langle H_p, K_\pm(p)\rangle,$$

for all $p \in \Sigma$, where H_p is the mean curvature vector at p. By these conventions, an ordinary sphere in a spatial "$t = 0$" section of Minkowski spacetime has $\theta_+\theta_- < 0$. Σ is a closed trapped surface (resp. a MOTS) if it is compact, two-sided and $\theta_\pm < 0$ (resp. $\theta_+ = 0$). Since the vector fields $K_\pm$ are unique up to rescaling by positive smooth functions, such (in)equalities do not depend on their particular choice. Of course, every compact two-sided maximal surface in (M, g) is a MOTS, but the converse does not necessarily hold.

Given any set $S \subset M$, let $\eta : [0, a) \to M$ be a nonspacelike curve starting at S. We say that η is an *S-ray* if the Lorentzian length of any segment of η starting at S up to any point p along the curve realizes the Lorentzian distance $d(S, p) \equiv \sup_{q\in S} d(q, p)$ from S to that point, where d denotes the Lorentzian distance function. It is easy to check that such an S-ray will have a reparametrization as a geodesic. In this paper, we will mostly consider null Σ-rays emanating from a surface Σ, in which case they have globally achronal images, are normal to Σ and have no focal points.

III. LIGHTLIKE SETS

In this section we display some basic results of a general theory of lighlike sets.

Definition III.1 *A non-empty set $A \subseteq M$ is said to be* future *[resp.* past*]* lightlike *if*

- *i) A is* locally achronal*, i.e., $\forall x \in A, \exists U \ni x$ open such that $A \cap U$ is achronal in $(U, g|_U)$;*
- *ii) For each $x \in A$ and for each $U \ni x$ neighbourhood of x for which $A \cap U$ is achronal in $(U, g|_U)$, there exists $y \in A\cap U$ such that $y \in J^+(x, U) \setminus \{x\}$ [resp. $y \in J^-(x, U) \setminus \{x\}$].*

A non-empty set is lightlike *if it is either past or future lightlike, or both.*

The following four propositions provide some important examples of lightlike sets. It is easy to see that each of these examples has a time-dual version which is a *past* lightlike set. Here and hereafter, we shall often state our results only in terms of future lightlike sets, in which case a time-dual version will be always be understood to hold.

Proposition III.1 *Let $\alpha : [a, b) \to M$ be a future-directed null geodesic ($-\infty < a < b \leq +\infty$), and $A \subset Im\alpha$ be any set dense in the image of α, i.e., such that $Im\alpha \subseteq \overline{A}$. Suppose that (M, g) is strongly causal. Then, A is a future lightlike set.*

Proposition III.2 *Let $C \subseteq M$ be any set. Then, $A := \partial I^-(C) \setminus \overline{C}$, if non-empty, is an achronal future lightlike set.*

Proposition III.3 *Let $S \subseteq M$ be any non-empty set and $\alpha : [0, a) \to M$ be a future-directed null S-ray. Then $A := \alpha[0, a)$ is an achronal future lightlike set.*

Proposition III.4 *If $S \subset M$ is a locally achronal null submanifold, then it is a (future and past) lightlike set.*

A lightlike set need not be a manifold. However, the following proposition, which can be seen as a partial converse to Proposition III.4, states that if it is a manifold, then it must be a *null* (a.k.a. lightlike) submanifold (i.e. an embedded C^∞ submanifold with everywhere degenerate induced metric), thus motivating its name.

Proposition III.5 *If a lightlike set $S \subset M$ is a submanifold, then it is a null submanifold.*

A *(future, past) lightlike C^k ($k \geq 0$) hypersurface* is simply a (future, past) lightlike set which is also a C^k hypersurface in M. These have a structure very similar to that of *achronal boundaries*, i.e., sets of the form $\partial I^{\pm}(C)$ for some set $C \subseteq M$, as the next proposition shows.

Proposition III.6 *Assume that S is a future lightlike C^0 hypersurface. Then:*

i) S is the union of (the images of) maximal future-directed null geodesics without future endpoints in S (when maximally extended in S, these will be called null generators *of S).*

ii) If α and β are two distinct null generators, then either $Im\alpha \cap Im\beta = \emptyset$ or $Im\alpha \cap Im\beta = \{p\}$, $p \in S$ being their common past endpoint.

iii) If S is achronal and $\gamma : [0, a) \to S$ is a null generator of S starting at $p = \gamma(0)$, then there exists no conjugate points to x_0 along γ.

iv) If S is closed, then all its null generators are future-inextendible (in M).

In the next section, it will be of importance to consider lightlike C^0 hypersurfaces whose null generators are *future-inextendible* (in M). Now, as Proposition III.2 indicates, natural examples of future lightlike C^0 hypersurfaces are sets of the form $\partial I^-(C) \setminus \overline{C}$. However, in this case the null generators are not in general future-inextendible, having future endpoints on $\overline{C}$. On the other hand, $\partial I^-(C)$ is (if non-empty) an achronal, closed C^0 hypersurface, though in general not a lightlike set, as already observed. But if it is, then, being a closed lightlike C^0 hypersurface, it does have future-inextendible null generators (by the item (iv) of the previous proposition). So it becomes pertinent to investigate situations where this does occur. To simplify the nomenclature, we adopt the following definition.

Definition III.2 *A future lightlike set $A \subseteq M$ is said to have a* future p-horizon *if $\partial I^-(A)$ is a non-empty future lightlike closed C^0 hypersurface (which will then have future-inextendible null generators by Prop. III.6 (iv)).* [1]

An important first example of a future lightlike set having a future p-horizon is as follows.

Proposition III.7 *Let $S \subseteq M$ be a non-empty set, and let $\alpha : [0, a) \to M$ be a future-inextendible null S-ray. Then $Im\alpha$ has a future p-horizon.*

IV. MAIN RESULTS

Our first main result is the following theorem, whose proof is adapted from a result by Eichemair, Galloway and Pollack [4]

Theorem IV.1 *Suppose (M, g) is globally hyperbolic, future null geodesically complete and satisfies the null convergence condition. Let $\Sigma \subset M$ be an acausal, future causally complete, maximal codimension 2 spacelike submanifold. Suppose furthermore that $\partial I^+(\Sigma) \setminus \Sigma$ contains a connected future lightlike set A with a future p-horizon. Then, the connected component of $\partial I^+(\Sigma) \setminus \Sigma$ containing A is a smooth totally geodesic null hypersurface with future-complete null geodesic generators.*

We consider two applications of this general result.

Let (M, g) is a *generalized plane wave*, i.e, $M = M_0 \times \mathbb{R}^2$ and

$$g(.,.) = g_0(.,.) + 2dudv + H(x, u)du^2, \quad (4.1)$$

[1] Again, the time-dual concept of having a *past p-horizon* applies to past lightlike sets. The name "p-horizon" comes from the standard concept of *particle horizon* in Relativity.

where g_0 a Riemannian metric on $M_0 = \mathbb{R}^{n-1}$, the variables (v, u) are the standard coordinates of $\mathbb{R}^2$, and $H : M_0 \times \mathbb{R} \to \mathbb{R}$ is a smooth real function. We have proven a special case of a conjecture presented in [3].

Theorem IV.2 *Assume that (M, g) is a 4-dimensional, vacuum (i.e., Ricci-flat) globally hyperbolic, future null geodesically complete generalized plane wave. Fix $(v_0, u_0) \in \mathbb{R}^2$, and let*

$$\Sigma = \Sigma_{v_0,u_0} = \{(x, v_0, u_0) \in M \mid x \in M_0\}.$$

Suppose further that

i) Σ is entirely contained in the causal future of some Cauchy hypersurface

ii) there exists a future-directed null Σ-ray $\eta : [0, \infty) \to M$ starting at Σ and not contained in the hypersurface $u = u_0$.

Then (M, g) is isometric to Minkowski spacetime.

The importance of *bifurcate Killing horizons* is that they appear in many exact solutions of the Einstein field equation in General Relativity, for instance as event horizons in the black hole spacetimes of the Kerr-Newman family or as the cosmological horizon in de Sitter spacetime. Indeed, it has been shown that stationary black hole event horizons are portions of bifurcate Killing horizons under much more general circumstances (see. e.g., [6]). They also are the natural arena for the the study of thermal properties of quantum states in semiclassical quantum gravity, such as the Hawking radiation of black holes and the Unruh effect. We now proceed to show, as a second application of our previous results, that a similar bifurcate structure appears around maximal codimension 2 compact acausal submanifolds in certain spacetimes *not necessarily* possessing Killing vector fields. Specifically, we have the following

Theorem IV.3 *Suppose that (M, g) is a globally hyperbolic, null geodesically complete spacetime satisfying the null convergence condition. Suppose that there exists a compact surface Σ such that*

i) For some Cauchy hypersurface V of (M, g), $\Sigma \subset V$, and $V \setminus \Sigma$ is the disjoint union of two connected C^0 hypersurfaces V_i, $i = 1, 2$, with non-compact closure,

ii) Σ is maximal in (M, g).

Then $\partial I^+(\Sigma) \setminus \Sigma$ [resp. $\partial I^-(\Sigma) \setminus \Sigma$] has exactly two connected components $\mathcal{H}_i^+$ $(i = 1, 2)$ [resp. $\mathcal{H}_i^-$ $(i = 1, 2)$] homeomorphic respectively to V_i $(i = 1, 2)$, so that $\Sigma = \cap_{i=1}^2 \overline{\mathcal{H}_i^+} \cap \overline{\mathcal{H}_i^-}$. Moreover, these connected components are smooth and totally geodesic null hypersurfaces in (M, g) with future-complete [resp. past-complete] null generators.

With a little more work, one can prove the following result.

Corollary IV.4 *Let (N, h) be an asymptotically flat Riemannian manifold of dimension $3 \leq n \leq 7$ with two asymptotically flat ends and zero scalar curvature. Suppose that (M, g) is the vacuum Cauchy development of (N, h) (viewed as a vacuum initial data set). Then, either (M, g) is null geodesically incomplete or it admits a bifurcate horizon.*

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