Cosymplectic p-spheres

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Abstract

We introduce cosymplectic circles and cosymplectic spheres and classify compact 3-manifolds that admit a cosymplectic circle. To any taut cosymplectic circle on a 3-manifold M we are able to canonically associate a complex structure and a conformal symplectic couple on $M \times \mathbb{R}$. We prove that a cosymplectic circle in dimension three is round if and only if it is taut. On the other hand, we provide examples in higher dimensions of cosymplectic circles which are taut but not round and examples of cosymplectic circles which are round but not taut.

1. Cosymplectic p-spheres

An *almost cosymplectic structure* on a manifold M is given by a pair (η, Ω), where η is a 1-form and Ω a 2-form on M such that $\eta \wedge \Omega^n$ is a volume form. Thus, in particular, $\dim(M) = 2n+1$. The almost cosymplectic structure (η, Ω) is said to be a *contact structure* if $d\eta = \Omega$ and a *cosymplectic structure* if η and Ω are both closed. On any almost cosymplectic manifold there exists a global vector field ξ uniquely determined by the conditions $i_\xi \eta = 1$, $i_\xi \Omega = 0$, called the *Reeb vector field*.

DEFINITION 1.1. *Let $(\eta_1, \Omega_1), \dots, (\eta_{p+1}, \Omega_{p+1})$ be $p+1$ almost cosymplectic structures on M. Consider the family $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda \in \mathbb{S}^p}$ where*

$$\eta_\lambda := \lambda_1 \eta_1 + \ldots + \lambda_{p+1} \eta_{p+1}, \quad \Omega_\lambda := \lambda_1 \Omega_1 + \ldots + \lambda_{p+1} \Omega_{p+1}.$$

If the pair $(\eta_\lambda, \Omega_\lambda)$ is an almost cosymplectic structure for every $\lambda = (\lambda_1, \dots, \lambda_{p+1}) \in \mathbb{S}^p$, then the family $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda \in \mathbb{S}^p}$ is called an almost cosymplectic p-sphere.

For $p = 1$ we will speak of an *almost cosymplectic circle*. For $p = 2$ we will simply speak of an *almost cosymplectic sphere*.

Next, we define two interesting types of almost cosymplectic p-spheres.

DEFINITION 1.2. *An (almost) cosymplectic p-sphere $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda \in \mathbb{S}^p}$ is said to be* taut *if all its elements give the same volume form, i.e., for any $\lambda, \lambda' \in \mathbb{S}^p$ we have*

$$\eta_\lambda \wedge (\Omega_\lambda)^n = \eta_{\lambda'} \wedge (\Omega_{\lambda'})^n.$$

DEFINITION 1.3. *An (almost) cosymplectic p-sphere $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda \in \mathbb{S}^p}$ is said to be* round *if for any $\lambda = (\lambda_1, \dots, \lambda_{p+1}) \in \mathbb{S}^p$ the vector field $\lambda_1 \xi_1 + \ldots + \lambda_{p+1} \xi_{p+1}$ is the Reeb vector field of the (almost) cosymplectic structure $(\eta_\lambda, \Omega_\lambda)$.*

Notice that Definition 1.1 generalizes the notion of *contact circles* and *contact spheres*, introduced by Geiges and Gonzalo [4], and the one of *contact p-spheres*, studied by Zessin [6].

EXAMPLE 1.4. *Let M be any orientable three-dimensional manifold and let η_1, η_2, η_3 be three 1-forms which are linearly independent at every point of M. Set*

$$\Omega_1 := \eta_2 \wedge \eta_3, \quad \Omega_2 := \eta_3 \wedge \eta_1, \quad \Omega_3 := \eta_1 \wedge \eta_2.$$

Then it is easy to see that (η_1, Ω_1), (η_2, Ω_2), (η_3, Ω_3) are almost cosymplectic structures. Moreover, one has

$$(\lambda_1 \eta_1 + \lambda_2 \eta_2 + \lambda_3 \eta_3) \wedge (\lambda_1 \Omega_1 + \lambda_2 \Omega_2 + \lambda_3 \Omega_3) = \eta_1 \wedge \eta_2 \wedge \eta_3 \neq 0.$$

Hence, (η_1, Ω_1), (η_2, Ω_2), (η_3, Ω_3) generate a taut almost cosymplectic sphere on M. Such an almost cosymplectic sphere is cosymplectic if the forms η_1, η_2, η_3 are closed. In particular, the above construction applies to the 3-torus $\mathbb{T}^3$ endowed with the 1-forms $\eta_\alpha := d\theta_\alpha$, $\alpha \in \{1,2,3\}$, where $\theta_1, \theta_2, \theta_3$ are the coordinates on $\mathbb{T}^3$.

Next we consider another special case of Example 1.4.

EXAMPLE 1.5. *Let H be the Heisenberg group, whose Lie algebra structure is given by*

$$[e_1, e_3] = 0, \quad [e_1, e_2] = \gamma e_3, \quad [e_3, e_2] = 0.$$

Let η_1, η_2, η_3 be the 1-forms dual to e_1, e_2, e_3, respectively. We have

$$\mathrm{d}\eta_1 = \mathrm{d}\eta_2 = 0, \qquad \mathrm{d}\eta_3 = -\frac{\gamma}{2} \eta_1 \wedge \eta_2.$$

Notice that the resulting sphere is neither contact nor cosymplectic, but (η_1, Ω_1) and (η_2, Ω_2) generate a left-invariant taut cosymplectic circle on H.

The following theorem generalizes a result of Zessin [6].

THEOREM 1.6. *Manifolds of dimension $4n+1$ do not admit any almost cosymplectic circle and thus any almost cosymplectic p-sphere for $p \geq 2$.*

Concerning the Reeb vector fields we have the following results.

PROPOSITION 1.7. *Let $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ be an almost cosymplectic circle. Then $i_{\xi_2} \Omega_1$ and $i_{\xi_1} \Omega_2$ nowhere vanish. As a consequence, the Reeb vector fields ξ_1 and ξ_2 of (η_1, Ω_1) and (η_2, Ω_2) are everywhere linearly independent.*

DEFINITION 1.8. *The* Reeb distribution *of an almost cosymplectic p-sphere $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda \in \mathbb{S}^p}$ is the distribution $\mathcal{V}$ generated by the Reeb vector fields of the generators, i.e. $\mathcal{V} = \langle \xi_1, \dots, \xi_{p+1} \rangle$.*

A natural question concerns the integrability of $\mathcal{V}$.

PROPOSITION 1.9. *Let $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ be a almost cosymplectic circle on a 3-manifold M. Then the Reeb distribution is given by $\mathcal{V} = \ker(i_{\xi_1} \Omega_2) = \ker(i_{\xi_2} \Omega_1)$. Hence $\mathcal{V}$ is integrable if and only if $i_{\xi_1} \Omega_2 \wedge \mathrm{d} i_{\xi_1} \Omega_2 = 0$, or equivalently if and only if $i_{\xi_2} \Omega_1 \wedge \mathrm{d} i_{\xi_2} \Omega_1 = 0$.*

Notice that an example of a taut cosymplectic circle on a 3-manifold with non-integrable Reeb distribution is given by Example 1.5 in the case $\gamma \neq 0$, where the condition $i_{\xi_1} \Omega_2 \wedge \mathrm{d} i_{\xi_1} \Omega_2 = 0$ of Proposition 1.9 is not satisfied.

Finally, we obtained a classification result in dimension three.

THEOREM 1.10. *Let M be a compact 3-dimensional manifold. M admits a cosymplectic circle if and only if M is either a 3-torus or a quotient of the Heisenberg group by a co-compact subgroup.*

2. More on taut and round

In this section we focus on round and taut cosymplectic p-spheres.

PROPOSITION 2.1. *Let $\{(\eta_1, \Omega_1), \dots, (\eta_{p+1}, \Omega_{p+1})\}$ be a cosymplectic p-sphere on a manifold M and let $\xi_1, \dots, \xi_{p+1}$ be the corresponding Reeb vector fields. Then the p-sphere is round if and only if*

(i) $\eta_i(\xi_j) + \eta_j(\xi_i) = 0$ *for any* $i, j \in \{1, \dots, p+1\}$, $i \neq j$

(ii) $i_{\xi_i} \Omega_j + i_{\xi_j} \Omega_i = 0$ *for any* $i, j \in \{1, \dots, p+1\}$.

Next we have a characterization of tautness in dimension 3.

LEMMA 2.2. *A cosymplectic circle $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ on a 3-manifold M is taut if it satisfies the conditions*

$$\eta_1 \wedge \Omega_1 = \eta_2 \wedge \Omega_2 \quad \text{and} \quad \eta_1 \wedge \Omega_2 = -\eta_2 \wedge \Omega_1.$$

THEOREM 2.3. *On a 3-manifold M a cosymplectic p-sphere is taut if and only if it is round.*

Theorem 2.3 does not hold in higher dimensions, as it is shown in the following example.

EXAMPLE 2.4. *Let us consider the cosymplectic structures (η_1, Ω_1), (η_2, Ω_2) on $\mathbb{T}^7$ given by $\eta_1 := dx_7$, $\eta_2 := dx_2$,*

$$\Omega_1 : = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6,$$
$$\Omega_2 : = dx_5 \wedge dx_4 - dx_3 \wedge dx_6 + (dx_1 + dx_3) \wedge dx_7.$$

One can prove that they generate a cosymplectic circle on $\mathbb{T}^7$ which is taut but not round, since the condition (ii) in Proposition 2.1 is not satisfied. On the other hand, the cosymplectic circle generated by $\eta_1 := dx_7$, $\eta_2 := -dx_2$,

$$\Omega_1 : = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6,$$
$$\Omega_2 : = dx_3 \wedge (dx_5 + dx_6) + dx_4 \wedge dx_5 + (dx_1 + dx_3) \wedge dx_6 + dx_1 \wedge dx_7,$$

is round but not taut.

To any taut cosymplectic circle $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ on a 3-manifold M we associate a complex structure J on $M \times \mathbb{R}$ in a canonical way. Indeed given a cosymplectic structure (η, Ω) on M we can define a symplectic structure on $M \times \mathbb{R}$ by $\omega := dt \wedge \eta + \Omega$. Recall that a pair of symplectic structures (ω_1, ω_2) on an oriented 4-manifold is said to be a *symplectic couple* [3] if one has $\omega_1 \wedge \omega_2 \equiv 0$ and ω_1^2, ω_2^2 are volume forms defining the positive orientation. A symplectic couple is called *conformal* if $\omega_1^2 = \omega_2^2$. Now, the cosymplectic structures (η_1, Ω_1) and (η_2, Ω_2) give rise to two symplectic structures ω_1 and ω_2 on $M \times \mathbb{R}$. They satisfy the following relations.

$$\omega_1^2 = 2dt \wedge \eta_1 \wedge \Omega_1, \quad \omega_2^2 = 2dt \wedge \eta_2 \wedge \Omega_2$$
$$\omega_1 \wedge \omega_2 = 2dt \wedge (\eta_1 \wedge \Omega_2 + \eta_2 \wedge \Omega_1).$$

As a consequence, we have $\omega_1 \wedge \omega_1 = \omega_2 \wedge \omega_2$ if and only if $\eta_1 \wedge \Omega_1 = \eta_2 \wedge \Omega_2$. Hence by Lemma 2.2 we have that (ω_1, ω_2) is a conformal symplectic couple on $M \times \mathbb{R}$ if and only if $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ is a taut cosymplectic circle on M. Therefore, by [3] to any taut cosymplectic circle $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ on a 3-dimensional smooth manifold M, we can associate a complex structure J on $M \times \mathbb{R}$. The obtained complex structure J is a recursion operator in the sense of [1], i. e., it is the unique endomorphism such that $i_X \omega_1 = i_{JX} \omega_2$ for any $X \in TM$.

3. Relation with 3-structures

In this section we show that 3-structures provide a class of examples of almost cosymplectic spheres on manifolds of dimension $4n+3$.

Recall that given an almost cosymplectic structure (η, Ω) there exist a Riemannian metric g (called compatible) and a tensor field ϕ such that $\phi^2 = -I + \eta \otimes \xi$, and $\Omega = g(-, \phi-)$. Then one can prove that $\eta \circ \phi = 0$, $\phi \xi = 0$ and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any $X, Y \in \Gamma(TM)$. The geometric structure (ϕ, ξ, η, g) is called *almost contact metric structure*. It follows that $g(X, \phi Y) = -g(\phi X, Y)$, so that the bilinear form $\Omega(X, Y) = g(X, \phi Y)$ is a 2-form, usually called the *fundamental 2-form*. Then one can prove that $\eta \wedge \Omega^n \neq 0$, where $\dim(M) = 2n+1$.

Now, when on the same manifold M there are given three distinct almost contact structures (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3) satisfying the following relations, for any even permutation (α, β, γ) of $\{1,2,3\}$,

$$\begin{aligned} \phi_\gamma = \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta, \\ \xi_\gamma = \phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha, \end{aligned} \tag{1}$$

we say that $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha \in \{1,2,3\}}$ is an *almost contact 3-structure* on M. One proves that the conditions (1) force the dimension of M to be necessarily $4n+3$ for some integer n and that there always exists a Riemannian metric g compatible with each almost contact structure and hence we can speak of *almost contact metric 3-structure*.

A 3-*quasi-Sasakian structure* is an almost contact metric 3-structure such that each structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is quasi-Sasakian, i.e. it is normal and the corresponding fundamental 2-form is closed. Remarkable subclasses of 3-quasi-Sasakian manifolds are 3-Sasakian and 3-cosymplectic manifolds.

THEOREM 3.1. *Let $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)_{\alpha \in \{1,2,3\}}$ be an almost contact metric 3-structure on M and let $\Omega_\alpha := g(\cdot, \phi_\alpha \cdot)$. Then, for any $\lambda := (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{S}^2$ the tensors $\phi_\lambda := \lambda_1 \phi_1 + \lambda_2 \phi_2 + \lambda_3 \phi_3$, $\xi_\lambda := \lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda_3 \xi_3$, and $\eta_\lambda := \lambda_1 \eta_1 + \lambda_2 \eta_2 + \lambda_3 \eta_3$, define an almost contact metric structure $(\phi_\lambda, \xi_\lambda, \eta_\lambda, g)$ on M. Moreover, $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2), (\eta_3, \Omega_3)\}$ generate an almost cosymplectic sphere on M which is both round and taut.*

In general the 1-forms η_1, η_2, η_3 can have different ranks even when they are constant. However, the following theorem holds.

THEOREM 3.2. *If $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)_{\alpha \in \{1,2,3\}}$ is a 3-quasi-Sasakian manifold of rank $4l+3$ then, for every $\lambda \in \mathbb{S}^2$, $(\phi_\lambda, \xi_\lambda, \eta_\lambda, g)$ defines a quasi-Sasakian structure of the same rank. In particular, $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2), (\eta_3, \Omega_3)\}$ generate an almost cosymplectic sphere where the 1-forms η_1, η_2, η_3 have the same rank $4l+3$.*

The following corollary improves the result of Zessin [6, Proposition 4] that any 3-Sasakian manifold admits a sphere of contact structures which is both taut and round.

COROLLARY 3.3. *Any 3-Sasakian manifold admits a sphere of Sasakian structures which is both taut and round. Similarly, any 3-cosymplectic manifold generates a cosymplectic sphere which is both taut and round.*

EXAMPLE 3.4. *Let N be a smooth manifold endowed with a hyperholomorphic symplectic structure in the sense of [1] (note that these geometric structures were also studied in [5], but with a different name). Namely, on N there are defined three symplectic structures ω_1, ω_2, ω_3 related to each other by means of the relations*

$$\omega_\alpha^\sharp \circ \omega_\beta^\flat = -\omega_\beta^\sharp \circ \omega_\alpha^\flat$$

for any $\alpha, \beta \in \{1,2,3\}$, $\alpha \neq \beta$. As a consequence, one can define three almost complex structures which satisfy the quaternionic identities. On $M := N \times \mathbb{R}^3$ we define for each even permutation (α, β, γ) of $\{1,2,3\}$

$$\eta_\alpha := dt_\alpha, \quad \Omega_\alpha := \omega_\alpha + \eta_\beta \wedge \eta_\gamma$$

where (t_1, t_2, t_3) are the global coordinates of $\mathbb{R}^3$. Then, one can prove that $\{(\eta_1, \Omega_1),\ (\eta_2, \Omega_2), (\eta_3, \Omega_3)\}$ generates a cosymplectic sphere on M.

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