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Riemannian connections with torsion adapted to almost CR structures

Giulia Dileo and Antonio Lotta

Dipartimento di Matematica - Università di Bari

giulia.dileo@uniba.it, antonio.lotta@uniba.it

Basic definitions

(M, HM, J, g) Riemannian almost CR manifold of type (n, k), $k \geq 0$:

- M real differentiable manifold of dimension $2n + k$;
- HM vector subbundle of the tangent bundle TM, of rank $2n$;
- $J : HM \to HM$ smooth fiber preserving bundle isomorphism such that $J^2 = -Id$;
- g compatible Riemannian metric:

$$g(X, Y) = g(JX, JY) \qquad X, Y \in \Gamma HM.$$

Definition We say that (M, HM, J, g) is a Riemannian almost CR manifold with torsion if there exists a metric connection on M with totally skew-symmetric torsion which parallelizes the structure (HM, J). Such a connection is called characteristic.

Some Examples:

- Almost Hermitian manifolds in the Gray-Hervella class $\mathcal{G}_1$ (e.g. nearly Kähler);
- Cosymplectic, Sasakian and quasi Sasakian manifolds [4];
- $\mathcal{S}$-manifolds and $\mathcal{K}$-manifolds (Blair [5]);
- Normal CR submanifolds $M \subset (N, J, g)$ of a Kähler manifold (Bejancu [3]);
- Naturally reductive homogeneous Riemannian CR manifolds.

Definition We define a quasi Sasakian CR manifold as a Riemannian almost CR manifold with torsion, having closed fundamental 2-form Φ, defined by

$$\Phi(X, Y) := g(X, JPY), \quad P : TM \to HM \text{ orthogonal projection.}$$

Characteristic connections

Let (M, HM, J, g) be a Riemannian almost CR manifold. Denoting by $P : TM \to HM$ and $Q : TM \to HM^\perp$ the orthogonal projections, we introduce the following tensors:

- Levi-Tanaka forms:

$$L : \Gamma HM \times \Gamma HM \to \Gamma HM^\perp \qquad L(X,Y) := Q[X,Y]$$
$$L' : \Gamma HM^\perp \times \Gamma HM^\perp \to \Gamma HM \qquad L'(\xi,\xi') := P[\xi,\xi']$$

- $N : \Gamma HM \times \Gamma HM \to \Gamma HM$, integrability tensor:

$$N(X,Y) := P([JX,JY] - [X,Y]) - JP([JX,Y] + [X,JY])$$

- $\theta_\xi : \Gamma HM \to \Gamma HM, \quad \xi \in \Gamma HM^\perp$

$$\theta_\xi(X) := P[\xi, JX] - JP[\xi, X]$$

- $\Gamma_X J : \Gamma HM \to \Gamma HM, \quad X \in \Gamma HM$

$$(\Gamma_X J)Y := \Gamma_X(JY) - J(\Gamma_X Y)$$

  where $\Gamma_X Y := P(\nabla^g_X Y)$.

Theorem Let (M, HM, J, g) be a Riemannian almost CR manifold. Then M admits a characteristic connection if and only if the following conditions are satisfied:

1) the tensor N is skew-symmetric,
2) $g(\theta_\xi(X), Y) = g([JX,Y] + [X,JY], \xi)$,
3) $(\mathcal{L}_\xi g)(X,Y) = 0$,
4) $(\mathcal{L}_X g)(\xi,\xi') = 0$,

for every $X, Y \in \Gamma HM$, and $\xi, \xi' \in \Gamma HM^\perp$.
Furthermore, the torsion of each characteristic connection satisfies:

$$T(X,Y,Z) = N(X,Y,Z) - \mathfrak{S}_{XYZ}\, g((\Gamma_{JX} J)Y, Z),$$
$$T(X,Y,\xi) = -g([X,Y],\xi) = -L_\xi(X,Y),$$
$$T(X,\xi,\xi') = -g([\xi,\xi'],X) = -L'_X(\xi,\xi').$$

This theorem generalizes known results of Friedrich and Ivanov, concerning the classes of almost Hermitian and almost contact metric manifolds [6].
In the case of partially integrable almost CR structures, condition 2) amounts to $\theta_\xi = 0$.

A construction on principal bundles

Let (M, HM, J, h) be a Riemannian almost CR manifold. We consider

$$\pi : Q \to M \text{ principal } G\text{-bundle}, \quad \omega : TQ \to \mathfrak{g} \text{ connection with curvature form } \Omega.$$

We denote by U^* the horizontal lift of a vector field $U \in \mathfrak{X}(M)$.
Q inherits natural Riemannian almost CR structures (HQ, J, g):

$$HQ := \pi_*^{-1}(HM) \subset Ker(\omega), \qquad JX^* := (JX)^* \quad X \in \Gamma HM,$$

$$g(U, W) = h(\pi_* U, \pi_* W) + \langle\, \omega(U), \omega(W)\,\rangle, \quad \text{(Jensen type metric)}$$

where $\langle\, , \rangle$ is a fixed inner product on $\mathfrak{g} := Lie(G)$.

Theorem (Q, HQ, J, g) admits a characteristic connection if and only if

1) (M, HM, J, h) admits a characteristic connection,

2) $\Omega(JX^*, JY^*) = \Omega(X^*, Y^*)$ for any $X, Y \in \Gamma HM$,

3) $\Omega(X^*, \xi^*) = 0$ for any $X \in \Gamma HM$ and $\xi \in \Gamma HM^\perp$.

Corollary Let (M, HM, J, h) be a Riemannian almost CR manifold with torsion. If M admits a characteristic connection with parallel torsion, then the frame bundle $L(M)$ admits a Riemannian almost CR structure with torsion.

The above construction gives rise to nontrivial flat examples of quasi Sasakian CR manifolds:

Theorem Let (N, J, h) be a Hermitian locally symmetric manifold with nonnegative sectional curvature at a point x. Then there exists a quasi Sasakian CR manifold M admitting a flat characteristic connection, fibering onto N, and with strongly pseudoconvex CR structure provided that the local de Rham decomposition of N at x contains no flat factor.

Theorem Let $(N, \varphi, \xi, \eta, g)$ be a locally φ-symmetric Sasakian manifold such that the sectional curvatures of 2-planes orthogonal to ξ_x at a point $x \in N$ are ≥ -3. Then there exists a quasi Sasakian CR manifold M admitting a flat characteristic connection and fibering onto N.

Indeed, in the first case (respectively in the second case) one can take $M := P(u)$, the holonomy bundle of the Levi-Civita (Tanaka-Webster) connection of N through an adapted frame u at x.

Some homogeneous models

Theorem Let $M = G/K$ be a homogeneous manifold, where G is a compact semisimple Lie group. Then the following conditions are equivalent:

a) M admits a G-invariant quasi Sasakian CR structure of CR dimension n.

b) M admits a G-invariant closed 2-form Φ of rank $2n$.

If b) holds, Φ is the fundamental 2-form associated to some G-invariant quasi Sasakian CR structure.

In fact, if a) or b) hold, M fibers over a flag manifold G/H, $K \subset H$, and the quasi Sasakian CR structure is obtained as a natural lift of a G-invariant Kähler structure (g_o, J) on G/H.

Example The (oriented) Stiefel manifolds $V_{k,n}(\mathbb{K})$, $k \geq 2$, admit several homogeneous quasi Sasakian CR structures.
If $\mathbb{K} = \mathbb{R}$ the manifold $V_{n,k}(\mathbb{R}) = SO(n)/SO(n-k)$, can be endowed with a rich family of structures, projecting onto flag manifolds of type:

$$N = SO(n)/U(n_1) \times \ldots \times U(n_p) \times U(1)^m \times SO(r).$$

If $\mathbb{K} = \mathbb{C}$, for $V_{n,k}(\mathbb{C}) = SU(n)/SU(n-k)$, one can choose as base manifold

$$N = SU(n)/S(U(n_1) \times \ldots \times U(n_p)).$$

If $\mathbb{K} = \mathbb{H}$, possible choices for N are of type

$$N = Sp(n)/U(n_1) \times \ldots \times U(n_p) \times U(1)^m \times Sp(r).$$

Example Let G be a compact semisimple Lie group and let $H \subset G$ be the centralizer of a torus $T \subset G$. Choose a G-invariant Kähler metric g_o with respect to the canonical complex structure J on the flag manifold G/H. Then (J, g_o) lifts to a left invariant, quasi Sasakian strongly pseudoconvex CR structure on G.

Example Every homogeneous manifold G/K where G is compact semisimple and K has non discrete center, carries a G-invariant quasi Sasakian CR structure.

Mixed sectional curvatures

Let (M, HM, J, g) be a Riemannian almost CR manifold. A sectional curvature $K^g(X, \xi)$ of a 2-plane spanned by unit vectors $X \in H_xM$ and $\xi \in H_xM^\perp$, $x \in M$, will be called a mixed sectional curvature.

Theorem Let (M, HM, J, g) be a Riemannian almost CR manifold with torsion, of type (n, k). Then

1) the mixed sectional curvatures are all nonnegative;

2) if the mixed sectional curvatures are all vanishing, HM is Levi flat and M is locally the Riemannian product of a $2n$-dimensional almost Hermitian manifold of type $\mathcal{G}_1$ and a k-dimensional Riemannian manifold.

The circumstance that all the mixed sectional curvatures are non null has a strong influence on the underlying almost CR structure:

Theorem Let (M, HM, J, g) be a Riemannian partially integrable almost CR manifold with torsion, of type (n, k). Assume that $HM^\perp$ is integrable and all the mixed sectional curvatures $K^g(X, \xi)$ at a point $x_o \in M$ are non null. Then, setting $n = (2a + 1)2^b$,

$$k \leq 2b + 1. \tag{1}$$

If $k \geq 2$, all the Levi forms $\mathcal{L}_\xi$ at x_o, $\xi \neq 0$, have signature $\left(\frac{n}{2}, \frac{n}{2}\right)$. Hence, (M, HM, J) is $\frac{n}{2}$-pseudoconcave at x_o.

Inequality (1) is sharp:

Example Fix a vector space V of $n \times n$ Hermitian matrices such that every nonzero $A \in V$ is nonsingular and V has the maximum dimension $2b+1$. Then one can construct a nilpotent graded Lie algebra of kind 2

$$\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}, \quad \mathfrak{m}_{-1} := \mathbb{C}^n, \quad \mathfrak{m}_{-2} := V^*, \qquad [X, Y](A) := \Im({}^t\bar{X}AY).$$

The simply connected Lie group M corresponding to $\mathfrak{m}$ is isomorphic to a CR quadric and can endowed with a left invariant Riemannian CR structure with torsion, of type $(n, 2b + 1)$. The space of the Levi forms $\mathcal{L}_\xi$, $\xi \in H_eM^\perp$, coincides with V and all the mixed sectional curvatures are non vanishing.

Remark In the above Theorem, in the case $k = 2$ one can drop the hypothesis that $HM^\perp$ is integrable.

Two global results

Theorem Let $N = G/H$ be an irreducible Hermitian symmetric space, where G is a compact simple Lie group. Let (J, g_o) be the G-invariant Kähler structure on N, where g_o is the normal metric.
Then (J, g_o) canonically lifts to a left invariant quasi Sasakian strongly pseudoconvex CR structure (HG, J, g) on G, projecting onto N and admitting a flat characteristic connection.
Furthermore, any simply connected, complete and irreducible quasi Sasakian CR manifold M admitting a flat characteristic connection arises by this construction, up to scaling the metric.

The proof relies on a classical result of Cartan and Schouten, implying that M is a compact simple Lie group or a 7-dimensional sphere [2]. We exclude here the latter possibility, owing to the CR integrability.

Theorem Let (M, HM, J, g) be a Riemannian almost CR manifold with torsion, of type (n, k), $k \geq 2$. Assume that M has constant sectional curvature c and M admits a characteristic connection with parallel torsion. If M is complete and simply connected, then $c = 0$.

The proof uses the classification of homogeneous CR structures in [7]. The assumption $\nabla T = 0$ is essential. Indeed, using a suitable parallelization, one can provide a Riemannian almost CR structure on S^7, whose all characteristic connections have non parallel torsion.

References

[1] I. Agricola: The Srní lectures on non-integrable geometries with torsion, *Arch. Math.(Brno)* **42** (2006).

[2] I. Agricola, T. Friedrich: A note on flat metric connections with antisymmetric torsion, *Differential Geom. Appl.* **28** (2010).

[3] A. Bejancu: Geometry of CR-submanifolds. D. Reidel Publishing Co., Dordrecht, 1986.

[4] D. E. Blair: The theory of quasi-Sasakian structures, *J. Differential Geom.* **1** (1967).

[5] D. E. Blair: Geometry of manifolds with structural group $\mathsf{U}(n) \times \mathsf{O}(s)$, *J. Differential Geom.* **4** (1976).

[6] T. Friedrich, S. Ivanov: Parallel spinors and connections with skew-symmetric torsion in string theory, *Asian J. Math.* **6** (2002).

[7] A. Krüger: Homogeneous Cauchy-Riemann structures, *Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4)* **18** (1991).