Sequences of harmonic maps in S^3

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XXIII International Fall Workshop on Geometry and Physics
Granada, 2014

Introduction

We define two transforms of a non-conformal harmonic map from a surface into the 3-sphere. Using these transforms one can in fact construct a sequence $\{f^p \mid p \in \mathbb{Z}\}$ of non-conformal harmonic maps. The transforms were inspired by the articles [3] and [1] and are generalisations of Lawson's polar construction for minimal surfaces in the 3-sphere (see [6]). Our motivation to investigate non-conformal harmonic maps is the study of almost complex surfaces in the nearly Kähler manifold $S^3 \times S^3$. We show how to associate an almost complex surface in the nearly Kähler manifold $S^3 \times S^3$ to a harmonic map in the 3-sphere. As a consequence one can associate to an almost complex surface in $S^3 \times S^3$ a sequence of almost complex surfaces. The results presented in this poster can be found in [5].

Harmonic maps into S^3

We identify the Euclidean space $\mathbb{R}^4$ with the ring of quaternions $\mathbb{H}$. The 3-sphere S^3 then is the set of unit quaternions $\{p \in \mathbb{H} \mid \|p\| = 1\}$. Elementary quaternion algebra shows that $p\alpha$ is orthogonal to p for every imaginary quaternion α. Therefore the tangent space at p is

$$T_pS^3 = \{p\alpha \mid \alpha \in \operatorname{Im}\mathbb{H}\}. \tag{1}$$

On a surface we will use complex coordinates, so in order to describe the complexified tangent vectors we need the complexified quaternions $\mathbb{H}\otimes\mathbb{C} = \mathbb{H}\oplus i\mathbb{H}$. The element i should not be considered as an element of $\mathbb{H}$. The complex bilinear extension of the Euclidean metric on $\mathbb{R}^4$ will be denoted by $\langle\ ,\ \rangle$. The conjugate of a complexified quaternion $p = p_1 + ip_2$ is equal to $\bar{p} = p_1 - ip_2$.

Now consider a non-conformal harmonic map $f\colon S \to S^3 \subset \mathbb{H}$ from a Riemann surface S into the 3-sphere S^3. Choose a local complex coordinate $z = x + iy$ on S and write $\partial := \frac{\partial}{\partial z}$ and $\bar\partial := \frac{\partial}{\partial \bar z}$. We introduce the $\mathbb{H}\otimes\mathbb{C}$-valued function

$$f_1 = \partial f.$$

Since $\langle f, f\rangle = 1$, it follows that $\langle f, f_1\rangle = 0$. Harmonicity means that $\partial\bar\partial f = -|\partial f|^2 f$. By the harmonicity of f, the function $\langle f_1, f_1\rangle$ is holomorphic. The map f is non-conformal, i.e. $\langle f_1, f_1\rangle$ is non-zero, so there exists a complex coordinate z, such that

$$\langle f_1, f_1\rangle = -1.$$

We will call such a coordinate an *adapted complex coordinate for* f. By (1) there exist functions α and β with values in $\operatorname{Im}\mathbb{H}$ such that $f_1 = \frac{1}{2}f(\alpha - i\beta)$. Then $\langle f_1, f_1\rangle = -1$ gives $\langle\alpha,\alpha\rangle - \langle\beta,\beta\rangle = -4$ and $\langle\alpha,\beta\rangle = 0$. Hence there is a non-negative smooth function ϕ such that $|\alpha| = 2\sinh\phi$ and $|\beta| = 2\cosh\phi$.

Note that α can be zero. We will work on the open subset of S where $\alpha \neq 0$. On this set we define the normal N as the real unit vector in the direction of $f(\alpha\times\beta)$. Now we have a complex moving frame $\mathcal{F} = \{f, f_1, \bar f_1, N\}$.

Lemma 1. *The moving frame equations for the frame $\mathcal{F}$ are*

$$\begin{aligned}
\partial f &= f_1,\\
\partial f_1 &= f + 2\partial\phi(\coth 2\phi\, f_1 + \operatorname{csch} 2\phi\, \bar f_1) + \mu\, N,\\
\partial \bar f_1 &= -\cosh 2\phi\, f,\\
\partial N &= -\mu \operatorname{csch} 2\phi(\operatorname{csch} 2\phi\, f_1 + \coth 2\phi\, \bar f_1),
\end{aligned} \tag{2}$$

where $\mu = \langle \partial f_1, N\rangle$. The compatibility conditions $\partial\bar\partial\mathcal{F} = \bar\partial\partial\mathcal{F}$ for $\mathcal{F}$ are

$$\begin{aligned}
2\partial\bar\partial\phi &= -\sinh 2\phi + |\mu|^2 \operatorname{csch} 2\phi,\\
\bar\partial\mu &= -2\bar\mu\partial\phi \operatorname{csch} 2\phi.
\end{aligned} \tag{3}$$

If f is a map into a great 2-sphere, then μ vanishes and the above compatibility condition for ϕ becomes the sinh-Gordon equation.

Sequences of harmonic maps

Now consider the transformations

$$\begin{aligned}
f^+ &= \frac{i}{2}\operatorname{sech}^2\phi(f_1 - \bar f_1) + \tanh\phi\, N,\\
f^- &= -\frac{i}{2}\operatorname{sech}^2\phi(f_1 - \bar f_1) + \tanh\phi\, N.
\end{aligned}$$

Note that if f would be conformal, that is, if $|\alpha| = |\beta|$, these transforms reduce to Lawson's polar surface N (see [6]). The following theorems can be proven by the moving frame equations (2) and the compatibility conditions (3) for the frame $\mathcal{F}$.

Theorem 1. *Let $f\colon S \to S^3$ be a non-conformal harmonic map from a Riemann surface S into the 3-sphere. Then the tranforms f^+ and f^- are also non-conformal harmonic maps from S to S^3. Furthermore, an adapted complex coordinate for f is also an adapted complex coordinate for f^+ and f^-.*

Theorem 2. *Let $f\colon S \to S^3$ be a non-conformal harmonic map. Then the $(+)$transform and $(-)$transform of f are mutual inverses in the sense that* $(f^+)^- = (f^-)^+ = f$.

By these theorems we can associate to a non-conformal harmonic map $f\colon S \to S^3$ a sequence $\{f^p \mid p \in \mathbb{Z}\}$ of such harmonic maps by defining $f^0 = f$ and, for every integer p, $f^{p+1} = (f^p)^+$ and $f^{p-1} = (f^p)^-$. Moreover z is an adapted complex coordinate for every map in the sequence.

Almost complex surfaces in $S^3 \times S^3$

The last result tells us that to a harmonic map into the 3-sphere we can associate an almost complex surface in the nearly Kähler manifold $S^3 \times S^3$. For the definitions and details on $S^3 \times S^3$ and its almost complex surfaces we refer to [2].

Theorem 3. *To a harmonic map from a simply connected surface into the 3-sphere S^3 one can associate an almost complex surface in $S^3 \times S^3$, and vice versa. Moreover the harmonic map is non-conformal if and only if the associated almost complex surface has non-vanishing holomorphic differential.*

Main idea of proof. Consider a harmonic map $f\colon S \to S^3$ with adapted complex coordinate $z = x + iy$. Then $\frac{\partial f}{\partial x} = f\alpha$ and $\frac{\partial f}{\partial y} = f\beta$. The integrability condition $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$ gives $\alpha_y - \beta_x = 2\alpha\times\beta$ and the harmonicity of f gives $\alpha_x + \beta_y = 0$. This harmonicity equation exactly is the integrability condition for the system of differential equations

$$X_x = -\beta, \qquad X_y = \alpha,$$

so this system has a $\mathbb{R}^3$-valued map X as a solution. The equation $\alpha_y - \beta_x = 2\alpha\times\beta$ now becomes

$$X_{xx} + X_{yy} = -2\, X_x \times X_y$$

By performing a dilation, the surface satisfies the Wente equation $X_{xx} + X_{yy} = -\frac{4}{\sqrt{3}} X_x \times X_y$. A correspondence theorem in [2] says that such a surface corresponds to an almost complex surface in $S^3 \times S^3$. □

In [4] Li, Ma and the authors proved an existence and uniqueness theorem for almost complex surfaces in $S^3 \times S^3$ with non-vanishing differential. The theorem states that such almost complex surfaces correspond to solutions ϕ and μ of the equations

$$\begin{aligned}
\partial\bar\partial\phi &= -\frac{1}{3}\sinh 2\phi + \frac{|\mu|^2}{3}\operatorname{csch} 2\phi,\\
\bar\partial\mu &= 2\bar\mu\partial\phi\operatorname{csch} 2\phi,
\end{aligned}$$

where ϕ is a positive real function and μ is a complex function. Up to an appropriate rescaling, these equations are equal to the compatibility conditions (3).

References

[1] M. Antic and L. Vrancken. Sequences of minimal surfaces in S^{2n+1}. *Israel J. Math.*, 179(1):493–508, 2010.

[2] J. Bolton, F. Dillen, B. Dioos, and L. Vrancken. Almost complex surfaces in the nearly Kähler $S^3 \times S^3$, 2013. To appear in Tohoku Math. J.

[3] J. Bolton and L. Vrancken. Transforms for minimal surfaces in the 5-sphere. *Differential Geom. Appl.*, 27:34–46, 2009.

[4] B. Dioos, H. Li, H. Ma, and L. Vrancken. Flat almost complex surfaces in $S^3 \times S^3$. Preprint, 2014.

[5] B. Dioos and L. Vrancken. Sequences of harmonic maps in the 3-sphere, 2014. Preprint. arXiv:1408.4414.

[6] H.B. Lawson. Complete minimal surfaces in S^3. *Ann. Math.*, 92:335–374, 1970.