Lorentzian Homogeneous gradient Ricci solitons: their Ricci tensor

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Aim

To describe the structure of the Ricci tensor on a locally homogeneous Lorentzian gradient Ricci soliton.

1. Definitions and general results

(M, g) is a pseudo-Riemannian manifold of dimension m	
Ricci solitons (M, g, X)	**Gradient Ricci solitons** (M, g, f)
M admits a smooth vector field X such that	There exists a potential function f on M such that
$\frac{1}{2}\mathcal{L}_X g + \rho = \lambda g$,	$\mathrm{Hess}_f + \rho = \lambda g$,
where λ is a constant and	where λ is a constant and
$\mathcal{L}_X$ is the Lie derivative for X.	Hess_f is the Hessian of f.
A Ricci soliton is said to be *shrinking* if $\lambda < 0$, *steady* if $\lambda = 0$ and *expanding* if $\lambda > 0$.	

Notation. We say that a gradient Ricci soliton is *non isotropic* if $\|\nabla f\|^2 \neq 0$ and *isotropic* if $\|\nabla f\|^2 = 0$.

Examples of Ricci solitons:

Einstein manifolds:
If (M, g) is Einstein and X is a Killing vector field, then the Ricci soliton equation is trivially satisfied.

Gaussian soliton:
A gradient Ricci soliton $(\mathbb{R}^n, g_0, f)$ where g_0 is the flat pseudo-Euclidean metric and $f(x) = \frac{\lambda}{2}\|x\|^2$.

Rigid (gradient) Ricci solitons:
A gradient Ricci soliton is *rigid* if it is isometric to a quotient of $N \times \mathbb{R}^k$, where N is an Einstein manifold and f is defined on the Euclidean factor as in the Gaussian soliton.

Lemma. Let (M, g, f) be a gradient Ricci soliton with constant scalar curvature.

1. We have the following relations:
   (a) $\mathrm{Ric}(\nabla f) = 0$.
   (b) $\|\nabla f\|^2 - 2\lambda f = \mathrm{const}$.
   (c) $R(X, Y, Z, \nabla f) = -(\nabla_X \rho)(Y, Z) - (\nabla_Y \rho)(X, Z)$.
   (d) $(\nabla_{\nabla f} \mathrm{Ric}) + \mathrm{Ric} \circ \mathcal{H}_f = R(\nabla f, \cdot)\nabla f$.
2. Let X be a Killing vector field, then
   (a) $\mathcal{L}_X(\mathrm{Hess}_f) = \mathrm{Hess}_{X(f)}$.
   (b) $\mathrm{grad}\{X(f)\}$ is a parallel vector field.
   (c) If $\lambda \neq 0$, then $\mathrm{grad}\{X(f)\} = 0$ if and only if $X(f) = 0$.
3. $\lambda((n+2)\lambda - \tau) = \|\mathrm{Hess}_f\|^2$.

Remark.
1. If (M, g, f) is isotropic and non-steady, then (M, g) is Einstein.
2. If (M, g, f) is steady, then $\|\mathrm{Hess}\, f\|^2 = 0$ and $\|\nabla f\|^2 = \mu$ is constant.

2. Non-steady case ($\lambda \neq 0$)

Theorem. Let (M, g, f) be a locally homogeneous Lorentzian non-steady gradient Ricci soliton. Then (M, g, f) locally splits as

$$(M, g, f) = (N_0 \times N_1 \times \mathbb{R}^k, g_0 + g_1 + g_e, f_0 + f_1 + f_e)$$

where (N_0, g_0, f_0) is an indecomposable locally homogeneous gradient Ricci soliton, (N_1, g_1) is an Einstein manifold with Einstein constant λ and $(\mathbb{R}^k, g_e, f_e)$ is pseudo-Euclidean space with $f_e(x) := \frac{\lambda}{2}\|x\|^2$. Some of the factors may not appear.

Remark. Note that if N_0 does not appear then the soliton is rigid.

Theorem. Let (M, g, f) be a locally homogeneous Lorentzian non-steady gradient Ricci soliton of dimension $m \leq 4$. Then (M, g, f) is rigid.

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3. Steady case ($\lambda = 0$)

Lemma.[3] A steady locally homogeneous Ricci soliton of dimension 2 either in the Riemannian or in the Lorentzian setting is flat.

Theorem. Let (M, g, f) be a locally homogeneous Lorentzian steady gradient Ricci soliton.

1. **If $\|\nabla f\|^2 < 0$, then (M, g) splits locally as a product $(\mathbb{R} \times N, -dt^2 + g_N)$, where (N, g_N) is a flat Riemannian manifold and f is orthogonal projection on $\mathbb{R}$.**
2. **If $\|\nabla f\|^2 = 0$, then one of the following two possibilities pertains:**
   (a) $\mathcal{H}_f = -\mathrm{Ric}$ has rank 2 and is 3-step nilpotent.
   (b) $\mathcal{H}_f = -\mathrm{Ric}$ has rank 1 and is 2-step nilpotent. In this case (M, g) is locally a strict Walker manifold.

4. Symmetric Lorentzian gradient Ricci solitons

We say that (N, g_N) is a *Cahen-Wallach symmetric space* if there are coordinates $(t, y, x_1, \dots, x_n)$ so:

$$g = 2\,dtdy + \left(\sum_{i=1}^{n} \kappa_i x_i^2\right) dy^2 + \sum_{i=1}^{n} dx_i^2 \text{ for } 0 \neq \kappa_i \in \mathbb{R}\,.$$

We shall always assume that all $\kappa_i \neq 0$ to ensure that (N, g_N) is indecomposable.

Theorem. Let (M, g, f) be a locally symmetric Lorentzian gradient Ricci soliton. Then (M, g) splits locally as a product $M = N \times \mathbb{R}^k$ where

1. **if (M, g, f) is not steady, then (N, g_N) is Einstein and the soliton is rigid,**
2. **if (M, g, f) is steady, then (N, g_N, f_N) is locally isometric to a Cahen-Wallach symmetric space.**

5. Three-dimensional locally homogeneous gradient Ricci solitons

Let (M, g) be a Lorentzian manifold of dimension 3. If (M, g) is strict Walker, i.e. admits a null parallel vector field, we may then (see, for example, [7]) find local adapted coordinates (t, x, y) so that

(1) $$g = 2dtdy + dx^2 + \phi(x, y)dy^2.$$

Theorem. Let (M, g, f) be a locally homogeneous Lorentzian gradient Ricci soliton of dimension 3. If (M, g, f) is non-trivial, then either it is rigid or it is a strict Walker manifold given by expression (1) where function ϕ can be:

1. **$\phi(x, y) = b^{-2}e^{bx}$ for $0 \neq b \in \mathbb{R}$, ∇f is spacelike.**
2. **$\phi(x, y) = \frac{1}{2}x^2\alpha(y)$ where $\alpha_y(y) = c\alpha^{3/2}(y)$ and $\alpha(y) > 0$ and ∇f is null.**
3. **$\phi(x, y) = \pm x^2$ define the Cahen-Wallach symmetric space and ∇f is null.**

In those three cases the soliton is steady.

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