# HIDDEN SYMMETRIES OF GEOMETRICAL STRUCTURES OF THE CLASSICAL PHASE SPACE OF GENERAL RELATIVISTIC TEST PARTICLE

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## Introduction

In literature as phase general relativistic infinitesimal symmetries (eventually *hidden symmetries*) are usually considered infinitesimal symmetries of the kinetic energy function on the cotangent bundle of the spacetime. It is very well known, [10], that such symmetries are given as the Hamiltonian lifts of functions *constant of motions*. Functions constant of motions which are polynomial on fibres of the cotangent bundle are given by Killing $k$–vector fields, $k \geq 1$. For $k = 1$ the corresponding infinitesimal symmetries are the flow lifts of Killing vector fields and so they are projectable on infinitesimal symmetries of the spacetime. For $k \geq 2$ the corresponding infinitesimal symmetries are not projectable and they are called *hidden symmetries*. Moreover, if we consider coupling with an electromagnetic 2–form conserved functions are generated by Killing-Maxwell multi–vector fields.

On the other hand the phase space of general relativistic test particle can be defined either as the *observer space*, [2], (a part of the unit pseudosphere bundle given by time-like future oriented vectors) or as the *1-jet space of motions*, [8]. The metric and an electromagnetic fields then define geometrical structures given by a 1–form and a closed 2–form. As phase infinitesimal symmetries we define infinitesimal symmetries of these forms. Phase infinitesimal symmetries which are projectable on the spacetime were studied on the observer space by Iwai [2] and on 1–jet space of motions by Janyška and Vitolo [8]. In both situations projectable symmetries are given by the flow lifts of Killing vector fields (eventually by Killing vector fieds which are infinitesimal symmetries of the electromagnetic field). In the paper we describe hidden (nonprojectable) infinitesimal symmetries for the phase space given as the 1-jet space of motions. We prove that hidden symmetries are given by the Hamilton–Jacobi lifts of (special) *conserved* phase functions and we give explicit construction of hidden symmetries generated by Killing (eventually Killing–Maxwell) multi–vector fields.

## 1. Phase space $T^*\boldsymbol{E}$

A *classical spacetime* is assumed to be an oriented and time oriented 4–dimensional manifold $\boldsymbol{E}$ equipped with a Lorentzian metric $g$ of signature $(1,3)$. We denote by $(x^\lambda)$ local coordinates on $\boldsymbol{E}$ and by $(x^\lambda, \dot{x}_\lambda)$ the induced fibred coordinates on $T^*\boldsymbol{E}$. In what follows we shall use notation $d^\lambda = dx^\lambda$, $\dot{d}_\lambda = d\dot{x}_\lambda$, $\partial_\lambda = \frac{\partial}{\partial x^\lambda}$ and $\dot{\partial}^\lambda = \frac{\partial}{\partial \dot{x}_\lambda}$ The inverse metric will be denoted by $\bar{g}$.

1.1. **Canonical symplectic structure.** Suppose the phase space to be the cotangent bundle $T^*\boldsymbol{E}$. Then we have the canonical symplectic 2–form $\Omega$ and the canonical Poisson 2-vector $\Lambda$ given by

$$\Omega = \dot{d}_\lambda \wedge d^\lambda \,, \qquad \Lambda = \dot{\partial}^\lambda \wedge \partial_\lambda \,.$$

Let us assume the *kinetic energy function*

$$H = \tfrac{1}{2} g^{\lambda\mu}\, \dot{x}_\lambda\, \dot{x}_\mu \,.$$

A function $K$ on $T^*\boldsymbol{E}$ is said to be a *constant of motion* if

$$0 = \{H, K\} = L_{X_H} K = -L_{X_K} H = g^{\lambda\mu}\, \dot{x}_\lambda\, \partial_\mu K - \tfrac{1}{2}\, \dot{\partial}^\rho K\, \partial_\rho g^{\lambda\mu}\, \dot{x}_\lambda\, \dot{x}_\mu\,. \tag{1.1}$$

**Remark 1.1.** A phase function $K$ is a constant of motions means that its Hamiltonian lift $X_K$ is an infinitesimal symmetry of the kinetic energy function or that $K$ is constant on geodesic curves since the Hamiltonian lift

$$X_H = g^{\lambda\rho}\, \dot{x}_\rho\, \partial_\lambda - \tfrac{1}{2}\, \partial_\lambda g^{\rho\sigma}\, \dot{x}_\rho\, \dot{x}_\sigma\, \dot{\partial}^\lambda\,. \tag{1.2}$$

is the tangent vector field of lifts of geodesics to $T^*\boldsymbol{E}$. □

Now let us discuss functions satisfying the equation (1.1). If $K$ is the pullback of a spacetime function then $K$ has to be a constant. Further suppose that $\overset{k}{K}$ is homogeneous of order $k$ on fibres of $T^*\boldsymbol{E}$, i.e.

$$\overset{k}{K} = \overset{k}{K}{}^{\lambda_1\dots\lambda_k}\, \dot{x}_{\lambda_1} \dots \dot{x}_{\lambda_k}\,, \qquad \overset{k}{K}{}^{\lambda_1\dots\lambda_k} \in C^\infty(\boldsymbol{E})\,.$$

Then $\overset{k}{K}$ can be considered as a symmetric $k$-vector field $\overset{k}{K} = \overset{k}{K}{}^{\lambda_1\dots\lambda_k}\, \partial_{\lambda_1} \odot \dots \odot \partial_{\lambda_k}$. Then the equation (1.1) is satisfied if and only if $\overset{k}{K}$ satisfies the *Killing equation*

$$[\bar{g}, \overset{k}{K}] = 0\,, \tag{1.3}$$

where $[\,,\,]$ is the Schouten bracket for symmetric multi-vector fields. So $\overset{k}{K}$, considered as a $k$–vector field, is a Killing tensor field.

**Remark 1.2.** For $k = 1$ we obtain that a vector field $\overset{1}{K}$ admits a symmetry of the kinetic energy function if and only if it is a Killing vector field. Moreover, the Hamiltonian lift of the corresponding function constant of motion is the flow lift $\mathcal{T}^*\overset{1}{K}$ of a vector field $\overset{1}{K}$ to the cotangent bundle. □

1.2. **(Souriau's) coupling with an electromagnetic field.** Let us consider a Maxwell (electromagnetic) field $F = F_{\lambda\mu}\, d^\lambda \wedge d^\mu$ satisfying the Maxwell equation $dF = 0$. Then we consider the "total" (joined) 2–form, [1],

$$\Omega^{\mathfrak{j}} = \Omega + \tfrac{1}{2} F = \dot{d}_\lambda \wedge d^\lambda + \tfrac{1}{2} F_{\lambda\mu}\, d^\lambda \wedge d^\mu\,.$$

We obtain the corresponding "total" (joined) Poisson 2–vector

$$\Lambda^{\mathfrak{j}} = \Lambda + \Lambda^{\mathfrak{e}} = \dot{\partial}^\lambda \wedge \partial_\lambda - \tfrac{1}{2} F_{\lambda\mu}\, \dot{\partial}^\lambda \wedge \dot{\partial}^\mu\,.$$

Now, assume a phase function $K$ satisfying

$$0 = \{H, K\}^{\mathfrak{j}} = L_{X^{\mathfrak{j}}_H} K = g^{\lambda\rho}\, \dot{x}_\rho\, \partial_\lambda K - \big(\tfrac{1}{2}\, \partial_\lambda g^{\rho\sigma}\, \dot{x}_\rho\, \dot{x}_\sigma + F_{\rho\lambda}\, g^{\rho\sigma}\, \dot{x}_\sigma\big)\, \dot{\partial}^\lambda K\,, \tag{1.4}$$

where $\{,\}^{\mathfrak{j}}$ is the "total" (joined) Poisson bracket.

According to [10] functions of the type

$$K = \overset{0}{K} + \sum_{k\ge 1} \overset{k}{K}{}^{\lambda_1\dots\lambda_k}\, \dot{x}_{\lambda_1} \dots \dot{x}_{\lambda_k}\,, \quad \overset{0}{K},\ \overset{k}{K}{}^{\lambda_1\dots\lambda_k} \in C^\infty(\boldsymbol{E})\,,$$

satisfy the equation (1.4) if and only if

$$0 = \sum_{k\ge 1} \big(\tfrac{1}{2}\, [\bar{g}, \overset{k-1}{K}]^{\sigma_1\dots\sigma_k} + k\, F_\rho{}^{\sigma_1}\, \overset{k}{K}{}^{\rho\sigma_2\dots\sigma_k}\big)\, \dot{x}_{\sigma_1} \dots \dot{x}_{\sigma_k}\,. \tag{1.5}$$

**Corollary 1.1.** *For a function* $K = \overset{0}{K} + \overset{1}{K}{}^{\lambda}\, \dot{x}_{\lambda}$ *two identities have to be satisfied*

$$0 = g^{\rho\sigma_1}\, \partial_\rho \overset{0}{K} + F_\rho{}^{\sigma_1}\, \overset{1}{K}{}^{\rho}\,, \qquad 0 = [\bar{g}\,, \overset{1}{K}]\,,$$

*which implies that* $\overset{1}{K}$ *is a Killing vector field and the identity* $d\overset{0}{K} + \overset{1}{K} \lrcorner F = 0$ *is satisfied. Then* $L_{\overset{1}{K}} F = 0$*. Moreover,*

$$X^{\mathrm{j}}_K = \overset{1}{K}{}^{\lambda}\, \partial_\lambda - (\partial_\lambda \overset{0}{K} + \partial_\lambda \overset{1}{K}{}^{\rho}\, \dot{x}_\rho + F_{\rho\lambda}\, \overset{1}{K}{}^{\rho})\, \dot{\partial}^{\lambda} = \overset{1}{K}{}^{\lambda}\, \partial_\lambda - \partial_\lambda \overset{1}{K}{}^{\rho}\, \dot{x}_\rho\, \dot{\partial}^{\lambda}$$

*which is the flow lift* $\mathcal{T}^* \overset{1}{K}$ *of the vector field* $\overset{1}{K}$*.* □

**Corollary 1.2.** *For a function function* $\overset{k}{K} = \overset{k}{K}{}^{\lambda_1 \dots \lambda_k}\, \dot{x}_{\lambda_1} \dots \dot{x}_{\lambda_k}$*,* $k \geq 2$*, we get*

$$0 = [\bar{g}, \overset{k}{K}]\,, \qquad 0 = F_\rho{}^{(\sigma_1}\, \overset{k}{K}{}^{\sigma_2 \dots \sigma_k)\rho}\,,$$

*i.e.* $\overset{k}{K}$ *is a Killing–Maxwell k–vector field,* [1]*. Moreover, the corresponding vector field is not projectable on spacetime and the infinitesimal symmetry is hidden.* □

## 2. Phase space as the 1st jet space of motions

Our theory is explicitly independent of scales, so we introduce the spaces of scales in the sense of [7]. Any tensor field carries explicit information on its scale dimension. We assume the following basic spaces of scales: the space of *time intervals* $\mathbb{T}$, the space of *lengths* $\mathbb{L}$ and the space of *mass* $\mathbb{M}$. We assume the *speed of light* $c \in \mathbb{T}^* \otimes \mathbb{L}$ and the *Planck constant* $\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$ as the *universal scales.* We denote as $u^0 \in \mathbb{T}^*$ a base.

2.1. **Classical phase space.** Now we assume the metric to be scaled, i.e. $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{E} \odot T^*\boldsymbol{E})$. A *spacetime chart* is defined to be a chart $(x^\lambda) \equiv (x^0, x^i) \in C^\infty(\boldsymbol{U}, \mathrm{I\!R} \times \mathrm{I\!R}^3)$, $\boldsymbol{U} \subset \boldsymbol{E}$ is open, of $\boldsymbol{E}$, which fits the orientation of spacetime and such that the vector field $\partial_0$ is timelike and time oriented and the vector fields $\partial_1, \partial_2, \partial_3$ are spacelike. Greek indices $\lambda, \mu, \dots$ will span spacetime coordinates, while Latin indices $i, j, \dots$ will span spacelike coordinates.

For a particle with a mass $m$ it is very convenient to use the *re-scaled metric* $G = \frac{m}{\hbar} g : \boldsymbol{E} \to \mathbb{T} \otimes (T^*\boldsymbol{E} \odot T^*\boldsymbol{E})$, $G^0_{\lambda\mu} = \frac{m}{\hbar_0}\, g_{\lambda\mu}$, and the associated contravariant re-scaled metric $\bar{G} = \frac{\hbar}{m} \bar{g} : \boldsymbol{E} \to \mathbb{T}^* \otimes (T\boldsymbol{E} \odot T\boldsymbol{E})$, $G^{\lambda\mu}_0 = \frac{\hbar_0}{m}\, g^{\lambda\mu}$.

We assume *time* to be a one-dimensional affine space $\boldsymbol{T}$ associated with the vector space $\bar{\mathbb{T}} = \mathbb{T} \otimes \mathbb{R}$. A *motion* is defined to be a 1–dimensional timelike submanifold $s : \boldsymbol{T} \hookrightarrow \boldsymbol{E}$. The *1st differential* of the motion $s$ is defined to be the tangent map $ds : T\boldsymbol{T} = \boldsymbol{T} \times \bar{\mathbb{T}} \to T\boldsymbol{E}$.

We assume as *phase space* the open subspace $\mathcal{J}_1\boldsymbol{E} \subset J_1(\boldsymbol{E}, 1)$ consisting of all 1–jets of motions. So elements of $\mathcal{J}_{1x}\boldsymbol{E}$ are classes of non-parametrized curves which have in a point $x \in \boldsymbol{E}$ the same tangent line lying inside the light cone, [5]. $\pi^1_0 : \mathcal{J}_1\boldsymbol{E} \to \boldsymbol{E}$ is a fibred manifold but NOT an affine bundle! The *velocity* of a motion $s$ is defined to be its 1–jet $j_1 s : \boldsymbol{T} \to \mathcal{J}_1(\boldsymbol{E}, 1)$. For each 1–dimensional submanifold $s : \boldsymbol{T} \hookrightarrow \boldsymbol{E}$ and for each $x \in \boldsymbol{T}$, we have $j_1 s(x) \in \mathcal{J}_1\boldsymbol{E}$ if and only if $ds(x)(u) \in T_{s(x)}\boldsymbol{E}$ is timelike, where $u \in \mathbb{T}$.

Any spacetime chart $(x^0, x^i)$ is related to each motion $s$ which means that $s$ can be locally expressed by $(x^0, x^i = s^i(x^0))$. Then we obtain the induced fibred coordinate chart $(x^0, x^i, x^i_0)$ on $\mathcal{J}_1\boldsymbol{E}$ such that $x^i_0 \circ s = \partial_0 s^i$. Moreover, there exists a time unit function $\boldsymbol{T} \to \mathbb{T}$ such that the 1st differential of $s$, considered as the map $ds : \boldsymbol{T} \to \bar{\mathbb{T}}^* \otimes T\boldsymbol{E}$, is normalized by $g(ds, ds) = -c^2$, for details see [5].

We shall always refer to the above fibred charts.

We define the *contact map* to be the unique fibred morphism $д : \mathcal{J}_1\boldsymbol{E} \to \bar{\mathbb{T}}^* \otimes T\boldsymbol{E}$ over $\boldsymbol{E}$, such that $д \circ j_1 s = ds$, for each motion $s$. We have $g\,(д, д) = -c^2$. The coordinate expression of $д$ is

$$д = c\,\alpha^0\,(\partial_0 + x^i_0\,\partial_i)\,, \qquad \text{where} \qquad \alpha^0 := 1/\sqrt{|g_{00} + 2\,g_{0j}\,x^j_0 + g_{ij}\,x^i_0\,x^j_0|}\,. \tag{2.1}$$

We define the *time form* to be the fibred morphism $\tau = -\frac{1}{c^2}\,g^\flat(д) : \mathcal{J}_1\boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$, considered as the scaled horizontal 1–form of $\mathcal{J}_1\boldsymbol{E}$. We have the coordinate expression

$$\tau = \tau_\lambda\,d^\lambda = -\tfrac{\alpha^0}{c}\,(g_{0\lambda} + g_{i\lambda}\,x^i_0)\,d^\lambda\,. \tag{2.2}$$

2.2. **Infinitesimal symmetries of the gravitational contact phase structure.** For a particle with mass $m$ we can unscale the time 1–form and obtain a contact 1–form $\widehat{\tau} = \frac{m\,c^2}{\hbar}\,\tau = \widehat{\tau}_\lambda\,d^\lambda$, where $\widehat{\tau}_\lambda = -\frac{m\,c\,\alpha^0}{\hbar}(g_{0\lambda} + g_{p\lambda}x^p_0)$. So the metric $g$ defines on the phase space $\mathcal{J}_1\boldsymbol{E}$ the *gravitational contact structure* $(-\widehat{\tau}, \Omega^{\mathfrak{g}})$, where $\Omega^{\mathfrak{g}} = -d\widehat{\tau}$. Then we have the dual Jacobi pair $(-\widehat{\gamma}^{\mathfrak{g}}, \Lambda^{\mathfrak{g}})$ given by the Reeb vector field $-\widehat{\gamma}^{\mathfrak{g}}$ and the 2–vector field $\Lambda^{\mathfrak{g}}$, [5].

We define an *infinitesimal symmetry of the gravitational contact phase structure* to be a phase vector field $X$ on $\mathcal{J}_1\boldsymbol{E}$ which is a symmetry of $\widehat{\tau}$, i.e. $L_X\widehat{\tau} = 0$. By naturality we have $L_X\Omega^{\mathfrak{g}} = 0$, $L_X\widehat{\gamma}^{\mathfrak{g}} = [X, \widehat{\gamma}^{\mathfrak{g}}] = 0$ and $L_X\Lambda^{\mathfrak{g}} = [X, \Lambda^{\mathfrak{g}}] = 0$. According to [4] any infinitesimal symmetry of the pair $(-\widehat{\tau}, \Omega^{\mathfrak{g}})$ is the Hamilton-Jacobi lift

$$X = d(\widehat{\tau}(\underline{X}))^\sharp + \widehat{\tau}(\underline{X})\,\widehat{\gamma}^{\mathfrak{g}} \tag{2.3}$$

of the phase function $\widehat{\tau}(\underline{X})$, where $\underline{X} = T\pi^1_0(X) : \mathcal{J}_1\boldsymbol{E} \to T\boldsymbol{E}$ is a generalized vector field in the sense of [9] such that $\widehat{\gamma}^{\mathfrak{g}}.(\widehat{\tau}(\underline{X})) = 0$. So, a generalized vector field $\underline{X}$ has to satisfy the following conditions:

1. (Projectability condition) The Hamilton-Jacobi lift (2.3) of the phase function $\widehat{\tau}(\underline{X})$ projects on $\underline{X}$.
2. (Conservation condition) The phase function $\widehat{\tau}(\underline{X})$ is *conserved*, i.e. $\widehat{\gamma}^{\mathfrak{g}}.(\widehat{\tau}(\underline{X})) = 0$ .

The following results were proved in [4].

**Theorem 2.1.** *Let $\underline{X} = \underline{X}^\lambda\,\partial_\lambda : \mathcal{J}_1\boldsymbol{E} \to T\boldsymbol{E}$, $\underline{X}^\lambda \in C^\infty(\mathcal{J}_1\boldsymbol{E})$, be a generalized vector field, then the following assertions are equivalent:*

*1. The Hamilton-Jacobi lift $X = d(\widehat{\tau}(\underline{X}))^\sharp + \widehat{\tau}(\underline{X})\,\widehat{\gamma}$ projects on $\underline{X}$.*

*2. The vertical prolongation*

$$V\underline{X} : V\mathcal{J}_1\boldsymbol{E} \to VT\boldsymbol{E} = T\boldsymbol{E} \oplus T\boldsymbol{E}$$

*has values in the kernel of $\widehat{\tau}$.*

*3. In coordinates*

$$(g_{0\rho} + g_{p\rho}\,x^p_0)\,\partial^0_j\underline{X}^\rho = 0\,. \tag{2.4}$$
□

**Lemma 2.2.** *For generalized vector fields $\underline{X}$ and $\underline{Y}$ satisfying the projectability condition we have*

$$\{\widehat{\tau}(\underline{X}), \widehat{\tau}(\underline{Y})\} + \widehat{\tau}(\underline{X})\,\widehat{\gamma}.(\widehat{\tau}(\underline{Y})) - \widehat{\tau}(\underline{Y})\,\widehat{\gamma}.(\widehat{\tau}(\underline{X})) = \widehat{\tau}(\underline{[X, Y]})\,. \tag{2.5}$$
□

**Remark 2.1.** Let us remark that on the right hand side of (2.5) there is the Jacobi bracket of functions $\widehat{\tau}(\underline{X})$ and $\widehat{\tau}(\underline{Y})$. □

**Theorem 2.3.** *Let $\underline{X}$ be a generalized vector field satisfying the projectability condition. Then the following assertions are equivalent:*

*1. The Hamilton-Jacobi lift $X = d(\widehat{\tau}(\underline{X}))^\sharp + \widehat{\tau}(\underline{X})\,\widehat{\gamma}$ is an infinitesimal symmetry of the gravitational contact phase structure.*

*2. The phase function $\widehat{\tau}(\underline{X})$ is conserved.*
*3. The vector field $[\widehat{\gamma}, X]$ is in $\ker \widehat{\tau}$.*
*4. In coordinates*

$$0 = g_{0\sigma}\,\partial_0 \underline{X}^\sigma + \tfrac{1}{2}\underline{X}^\sigma\,\partial_\sigma g_{00} + \big(g_{p\sigma}\,\partial_0 \underline{X}^\sigma + g_{0\sigma}\,\partial_p \underline{X}^\sigma + \underline{X}^\sigma\,\partial_\sigma g_{0p}\big)\,x_0^p + \big(g_{p\sigma}\,\partial_q \underline{X}^\sigma + \underline{X}^\sigma\,\partial_\sigma g_{pq}\big)\,x_0^p\,x_0^q\,. \tag{2.6}$$
□

**Corollary 2.4.** *For generalized vector fields $\underline{X}$ and $\underline{Y}$ satisfying the projectability and the conservation conditions we have*

$$\{\widehat{\tau}(\underline{X}), \widehat{\tau}(\underline{Y})\} = \widehat{\tau}(\underline{[X, Y]})\,. \tag{2.7}$$

*Moreover, the phase function $\widehat{\tau}(\underline{[X, Y]})$ is conserved.* □

**Theorem 2.5.** *The Lie algebra of infinitesimal symmetries of the gravitational contact phase structure are the Hamilton-Jacobi lifts of phase functions $\widehat{\tau}(\underline{X})$, where generalized vector fields $\underline{X}$ satisfy the projectability and the conservation conditions.*

*Moreover, if $\underline{X}$ factorises through a spacetime vector field, then the corresponding infinitesimal symmetry is projectable. If $\underline{X}$ is a generalized vector field which is not factorisable through a spacetime vector field, then the corresponding infinitesimal symmetry is hidden.* □

2.3. **Infinitesimal symmetries of the gravitational contact phase structure and Killing multi–vector fields.** In [4] it was proved that a symmetric $k$-vector field $\overset{k}{K}$, $k \geq 1$, admits generalized vector field satisfying the projectability condition. Such generalized vector fields are given by

$$\underline{X}[\overset{k}{K}] = k\,\underbrace{\widehat{\tau} \lrcorner \ldots \lrcorner \widehat{\tau} \lrcorner}_{(k-1)-times}\overset{k}{K} - (k-1)\overset{k}{K}(\widehat{\tau}, \ldots, \widehat{\tau})\,\widehat{д} : \mathcal{J}_1\boldsymbol{E} \to T\boldsymbol{E}\,, \tag{2.8}$$

where $\widehat{д} = \frac{\hbar}{m\,c^2}\,д$. Then we obtain the induced phase function

$$\widehat{\tau}\big(\underline{X}[\overset{k}{K}]\big) = \overset{k}{K}(\widehat{\tau}) := \overset{k}{K}(\widehat{\tau}, \ldots, \widehat{\tau}) = \overset{k}{K}{}^{\lambda_1\ldots\lambda_k}\,\widehat{\tau}_{\lambda_1}\ldots\widehat{\tau}_{\lambda_k}\,.$$

**Theorem 2.6.** *The phase function $\overset{k}{K}(\widehat{\tau})$ is conserved with respect to the gravitational Reeb vector field, i.e. $\widehat{\gamma}^{\mathfrak{g}}.\overset{k}{K}(\widehat{\tau}) = 0$, if and only if $\overset{k}{K}$ is a Killing $k$–vector field.* □

**Remark 2.2.** Let $\overset{0}{K}$ be a spacetime function. Then $\widehat{\gamma}^{\mathfrak{g}}.\overset{0}{K} = 0$ if and only if $\overset{0}{K}$ is a constant.

**Corollary 2.7.** *The Hamilton–Jacobi lift of a phase function $\overset{k}{K}(\widehat{\tau})$, $k \geq 1$, is an infinitesimal symmetry of the gravitational contact phase structure $(-\widehat{\tau}, \Omega^{\mathfrak{g}})$ if and only if $\overset{k}{K}$ is a Killing $k$–vector field. Moreover, for $k = 1$ the corresponding infinitesimal symmetry coincides with the jet flow lift and is projectable on spacetime. For $k \geq 2$ the corresponding infinitesimal symmetry is hidden.* □

**Remark 2.3.** For a constant $\overset{0}{K}$ and Killing $k$-vector fields $\overset{k}{K}$, $k \geq 1$, the conserved phase function of the type

$$K = \overset{0}{K} + \sum_{k\geq 1}\overset{k}{K}(\widehat{\tau}) \tag{2.9}$$

admits the infinitesimal symmetry

$$X[K] = \overset{0}{K}\,\widehat{\gamma} + \sum_{k\geq 1} X[\overset{k}{K}]$$

which projects on the generalized vector field

$$\underline{X}[K] = \big(\overset{0}{K} - \sum_{k\geq 2}(k-1)\,\overset{k}{K}(\widehat{\tau})\big)\,\widehat{\boldsymbol{\imath}} + \overset{1}{K} + \sum_{k\geq 2} k\,\underbrace{\widehat{\tau}\,\lrcorner\,\dots\,\lrcorner\,\widehat{\tau}\,\lrcorner}_{(k-1)-times}\overset{k}{K}$$

satisfying the projectability and the conservation conditions. □

2.4. **Coupling with an electromagnetic field.** Further, we assume an *electromagnetic field* to be a closed scaled 2-form on $\boldsymbol{E}$

(2.10)
$$F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \bigwedge^2 T^*\boldsymbol{E}.$$

Given a particle with a *charge* $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$ the rescaled electromagnetic field $\widehat{F} = \frac{q}{\hbar} F$ can be incorporated into the geometrical structure of the phase space, i.e. the gravitational form. Namely, we define the *joined (total) phase 2-form*

(2.11)
$$\Omega^{\mathfrak{j}} := \Omega^{\mathfrak{g}} + \Omega^{\mathfrak{e}} = \Omega^{\mathfrak{g}} + \tfrac{1}{2}\widehat{F} : \mathcal{J}_1\boldsymbol{E} \to \bigwedge^2 T^*\mathcal{J}_1\boldsymbol{E}.$$

The pair $(-\widehat{\tau}, \Omega^{\mathfrak{j}})$ is almost–cosymplectic–contact, i.e. it is regular and both forms are closed, [6]. Then the dual almost–coPoisson–Jacobi pair is $(-\widehat{\gamma}^{\mathfrak{j}}, \Lambda^{\mathfrak{j}})$, $\widehat{\gamma}^{\mathfrak{j}} = \frac{\hbar}{m\,c^2}(\gamma^{\mathfrak{g}} + \gamma^{\mathfrak{e}})$, where

(2.12)
$$\gamma^e : \mathcal{J}_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\mathcal{J}_1\boldsymbol{E}$$

with the coordinate expression

$$\gamma^e = -(G^{i\lambda}_0 - x^i_0\, G^{0\lambda}_0)\,(\widehat{F}_{0\lambda} + \widehat{F}_{j\lambda}x^j_0)\, u^0 \otimes \partial^0_i.$$

$\gamma^e$ is the *Lorentz force* associated with $F$. Further, $\Lambda^{\mathfrak{j}} = \Lambda^{\mathfrak{g}} + \Lambda^{\mathfrak{e}}$, where

$$\Lambda^{\mathfrak{e}} = \tfrac{1}{2\,(c_0\alpha^0)^2}(G^{i\lambda}_0 - x^i_0\, G^{0\lambda}_0)\,(G^{j\mu}_0 - x^j_0\, G^{0\mu}_0)\,\widehat{F}_{\lambda\mu}\,\partial^0_i \wedge \partial^0_j\,.$$

We define a *phase infinitesimal symmetry* to be a vector field $X$ on $\mathcal{J}_1\boldsymbol{E}$ such that: (1) $L_X\widehat{\tau} = 0$; (2) $L_X\Omega^{\mathfrak{j}} = 0$.

**Remark 2.4.** The conditions (1) and (2) are equivalent to $[X, \widehat{\gamma}^{\mathfrak{j}}] = 0$ and $[X, \Lambda^{\mathfrak{j}}] = 0$. □

**Lemma 2.8.** *A phase vector field $X$ is an infinitesimal symmetry of $\Omega^{\mathfrak{j}}$ if and only if it is of the form*

$$X = df^{\sharp\mathfrak{j}} + h\,\widehat{\gamma}^{\mathfrak{j}}\,,$$

*where $f$ is a conserved phase function, i.e. $\widehat{\gamma}^{\mathfrak{j}}.f = 0$, and $h = \widehat{\tau}(\underline{X})$, $\underline{X} = T\pi^1_0(X)$.*

*Proof.* We have the splitting $T\mathcal{J}_1\boldsymbol{E} = \ker\widehat{\tau} \oplus \langle\widehat{\gamma}^{\mathfrak{j}}\rangle$, i.e. $X = \widetilde{X} + h\,\widehat{\gamma}^{\mathfrak{j}}$, where $\widehat{\tau}(\widetilde{X}) = 0$ and $h$ is a phase function. Then from $\widehat{\tau}(\widehat{\gamma}^{\mathfrak{j}}) = 1$ we have $h = \widehat{\tau}(\underline{X})$.

Further the phase 2–form $\Omega^{\mathfrak{j}}$ is closed, then from $i_{\widehat{\gamma}^{\mathfrak{j}}}\Omega^{\mathfrak{j}} = 0$ we obtain $0 = L_X\Omega^{\mathfrak{j}} = d i_{\widetilde{X}}\Omega^{\mathfrak{j}}$, which implies locally that $i_{\widetilde{X}}\Omega^{\mathfrak{j}} = df$ for a phase function $f$, i.e. $\widetilde{X} = df^{\sharp\mathfrak{j}}$. Moreover, $\widehat{\gamma}^{\mathfrak{j}}.f = i_{\widehat{\gamma}^{\mathfrak{j}}}df = i_{\widehat{\gamma}^{\mathfrak{j}}} i_{\widetilde{X}}\Omega^{\mathfrak{j}} = -i_{\widetilde{X}} i_{\widehat{\gamma}^{\mathfrak{j}}}\Omega^{\mathfrak{j}} = 0$. □

**Lemma 2.9.** *A phase vector field*

$$X = df^{\sharp\mathfrak{j}} + \widehat{\tau}(\underline{X})\,\widehat{\gamma}^{\mathfrak{j}}$$

*is an infinitesimal symmetry of $\widehat{\tau}$ if and only if $f$ is of the form*

$$f = \widehat{\tau}(\underline{X}) + \breve{f}$$

*for a generalized vector field* $\underline{X}$ *and a spacetime function* $\breve{f} \in C^\infty(\boldsymbol{E})$ *such that*

$$d\breve{f} = \underline{X} \lrcorner \widehat{F} \,. \tag{2.13}$$

*Proof.* It can be proved in coordinates. □

**Theorem 2.10.** *All phase infinitesimal symmetries of the total phase structure are vector fields of the type*

$$X = d(\widehat{\tau}(\underline{X}) + \breve{f})^{\sharp\mathfrak{j}} + \widehat{\tau}(\underline{X})\, \widehat{\gamma}^{\mathfrak{j}} \tag{2.14}$$

*where* $\underline{X}$ *is a generalized vector field and* $\breve{f} \in C^\infty(\boldsymbol{E})$ *satisfying the following conditions:*

*1)* $d\breve{f} = \underline{X} \lrcorner \widehat{F}$ *.*

*2) (Projectability) The vector field* (2.14) *projects on* $\underline{X}$*.*

*3) (Conservation) Phase function* $\widehat{\tau}(\underline{X}) + \breve{f}$ *is conserved, i.e.* $\widehat{\gamma}^{\mathfrak{j}}.(\widehat{\tau}(\underline{X}) + \breve{f}) = 0$*.*

*Proof.* It follows from Lemmas 2.8 and 2.9. □

**Remark 2.5.** Let us note that to find a pair $(\underline{X}, \breve{f})$ satisfying the projectability condition 2) of the above Theorem, it is sufficient to find a generalized vector field satisfying Theorem 2.6. It follows from the fact that $d\breve{f}^{\sharp\mathfrak{j}}$ is a vertical vector field.

Further, if the condition 1) and 2) are satisfied then the conservation condition $\widehat{\gamma}^{\mathfrak{j}}.(\widehat{\tau}(\underline{X}) + \breve{f}) = 0$ is equivalent with the conservation condition given in Theorem 2.3 which follows from $\widehat{\gamma}^{\mathfrak{g}}.\breve{f} = -\widehat{\gamma}^{\mathfrak{e}}.\widehat{\tau}(\underline{X})$. □

Now, let us consider a phase function

$$K = \overset{0}{K} + \sum_{k\geq 1} \overset{k}{K}(\widehat{\tau})\,, \quad \overset{0}{K} \in C^\infty(\boldsymbol{E})\,, \tag{2.15}$$

i.e. we consider $\breve{f} = \overset{0}{K}$ a spacetime function and symmetric multi–vector fields $\overset{k}{K}$, $k \geq 1$. Since the electromagnetic part of the joined structure admits only vertical vector fields we obtain the generalized vector field

$$\underline{X}[K] = T\pi^1_0\big(dK^{\sharp\mathfrak{j}} + \sum_{k\geq 1} \overset{k}{K}(\widehat{\tau})\,\widehat{\gamma}^{\mathfrak{j}}\big) = \overset{1}{K} - \sum_{k\geq 2}(k-1)\,\overset{k}{K}(\widehat{\tau})\,\daleth + \sum_{k\geq 2} k\,\underbrace{\widehat{\tau} \lrcorner \dots \lrcorner \widehat{\tau} \lrcorner}_{(k-1)-times} \overset{k}{K}\,.$$

Such generalized vector field satisfies the projectability condition and we have to find conditions for the function (2.15) to be conserved with respect to the joined Reeb vector field, i.e. $\widehat{\gamma}^{\mathfrak{j}}.K = 0$.

**Lemma 2.11.** *We have*

$$\widehat{\gamma}^{\mathfrak{e}}.\overset{0}{K} = 0\,,$$

$$\widehat{\gamma}^{\mathfrak{e}}.\overset{k}{K}(\widehat{\tau}) = -k\,\frac{\hbar^2}{m^2\,c^2}\,\overset{k}{K}{}^{\rho\lambda_1\dots\lambda_{k-1}}\,\widehat{F}^{\lambda_k}{}_{\rho}\,\widehat{\tau}_{\lambda_1}\dots\widehat{\tau}_{\lambda_k}\,, \quad k \geq 1\,,$$

*where* $\widehat{F}^{\lambda_k}{}_{\rho} = g^{\sigma\lambda_1}\,\widehat{F}_{\sigma\rho}$*.*

*Proof.* We have

$$\widehat{\gamma}^{\mathfrak{e}}.\widehat{\tau}_\rho = \frac{\hbar}{m\,c}\,(\widehat{F}_{0\rho} + \widehat{F}_{p\rho}\,x^p_0)\,\alpha^0$$

which implies

$$\widehat{\gamma}^{\mathfrak{e}}.\overset{k}{K}(\widehat{\tau}) = k\,\overset{k}{K}{}^{\rho\lambda_1\dots\lambda_{k-1}}\,(\widehat{\gamma}^{\mathfrak{e}}.\widehat{\tau}_\rho)\,\widehat{\tau}_{\lambda_1}\dots\widehat{\tau}_{\lambda_{k-1}} = -k\,\frac{\hbar^2}{m^2\,c^2}\,\overset{k}{K}{}^{\rho\lambda_1\dots\lambda_{k-1}}\,\widehat{F}_{\sigma\rho}\,g^{\sigma\lambda_k}\,\widehat{\tau}_{\lambda_1}\dots\widehat{\tau}_{\lambda_k}\,.$$ □

**Theorem 2.12.** *A phase function* (2.15) *is conserved with respect to the joined Reeb vector field, i.e.* $\widehat{\gamma}^{\mathfrak{j}}.K = 0$*, if and only if*

$$g^{\rho\lambda}\,\partial_\rho \overset{0}{K} + \overset{1}{K}{}^{\rho}\,\widehat{F}^{\lambda}{}_{\rho} = 0\,, \tag{2.16}$$

$$\nabla^{(\lambda_1} \overset{k}{K}{}^{\lambda_2\dots\lambda_{k+1})} + (k+1)\,\overset{k+1}{K}{}^{\rho(\lambda_1\dots\lambda_k}\,\widehat{F}^{\lambda_{k+1})}{}_{\rho} = 0\,, \tag{2.17}$$

*for* $k = 1, 2, \dots$ *.*

*Proof.* From Lemma 2.11 we have

$$\widehat{\gamma}^{\mathfrak{j}}.K = \widehat{\gamma}^{\mathfrak{g}}.K + \widehat{\gamma}^{\mathfrak{e}}.K = -\frac{\hbar^2}{m^2\,c^2}\,\Big[\big(g^{\rho\lambda}\,\partial_\rho \overset{0}{K} + \overset{1}{K}{}^{\rho}\,\widehat{F}^{\lambda}{}_{\rho}\big)\,\widehat{\tau}_{\lambda_1}$$
$$+ \sum_{k=1} \big(\tfrac{1}{2}[\bar{g}, \overset{k}{K}]^{\lambda_1\dots\lambda_{k+1}} + (k+1)\,\overset{k+1}{K}{}^{\rho\lambda_1\dots\lambda_k}\,\widehat{F}^{\lambda_{k+1}}{}_{\rho}\big)\,\widehat{\tau}_{\lambda_1}\dots\widehat{\tau}_{\lambda_{k+1}}\Big]\,.$$

From

$$\nabla^{(\lambda_1} \overset{k}{K}{}^{\lambda_2\dots\lambda_{k+1})} = \tfrac{1}{2}[\bar{g}, \overset{k}{K}]^{\lambda_1\dots\lambda_{k+1}}$$

we obtain Theorem. □

**Corollary 2.13.** *The vector field* $X[K] = dK^{\sharp\mathfrak{j}} + h\,\widehat{\gamma}^{\mathfrak{j}}$*, where the phase function* $K$ *is given by* (2.15) *and* $h = K - \overset{0}{K}$*, is an infinitesimal symmetry of the almost-cosymplectic-contact pair* $(-\widehat{\tau}, \Omega^{\mathfrak{j}})$ *if and only if the conditions* (2.16) *and* (2.17) *are satisfied and* $d\overset{0}{K} = \underline{X}[K] \,\lrcorner\, \widehat{F}$*.* □

**Corollary 2.14.** *Let us assume the a (special) phase function* $K = \overset{0}{K} + \overset{1}{K}(\widehat{\tau})$*. Then the conditions* (2.16) *and* (2.17) *are reduced to*

$$\partial_\rho \overset{0}{K} - \overset{1}{K}{}^{\sigma}\,\widehat{F}_{\sigma\rho} = 0\,, \qquad \nabla^{(\lambda_1} \overset{1}{K}{}^{\lambda_2)} = 0$$

*and we obtain the result of* [8]*, i.e.* $\overset{1}{K}$ *is a Killing vector field and* $\overset{0}{K}$ *and* $\overset{1}{K}$ *are related by the formula* $d\overset{0}{K} = \overset{1}{K} \,\lrcorner\, \widehat{F}$ *which implies that* $\overset{1}{K}$ *is an infinitesimal symmetry of* $\widehat{F}$*. Moreover, the corresponding infinitesimal symmetry is the flow lift* $\mathfrak{J}_1\overset{1}{K}$ *which projects on* $\overset{1}{K}$*.* □

**Corollary 2.15.** *If we assume a phase function* $K$ *given by* (2.15) *for* $k \geq 2$ *then the conditions* (2.16) *and* (2.17) *are satisfied if* $d\overset{0}{K} = \overset{1}{K} \,\lrcorner\, \widehat{F}$*,* $\overset{1}{K}$ *is a Killing vector field which is an infinitesimal symmetry of* $\widehat{F}$ *and* $\overset{2}{K}, \dots, \overset{k}{K}$ *are Killing–Maxwell multi–vector fields. Moreover, the condition* $d\overset{0}{K} = \underline{X}[K] \,\lrcorner\, \widehat{F}$ *leads to the condition*

$$0 = \sum_{k\geq 2} \big((k-1)K(\widehat{\tau})\,\text{Д} \,\lrcorner\, \widehat{F} - k\,(\underbrace{\widehat{\tau} \,\lrcorner\, \dots \,\lrcorner\, \widehat{\tau} \,\lrcorner}_{(k-1)-times}\overset{k}{K}) \,\lrcorner\, \widehat{F}.\big) \tag{2.18}$$

*The corresponding infinitesimal symmetry is hidden.* □

**Remark 2.6.** Let us assume a phase function $K = \overset{2}{K}(\widehat{\tau})$. Then the condition (2.18) has the coordinate expression

$$\overset{2}{K}{}^{\rho\lambda}\widehat{F}_{\rho\alpha} = 0\,, \quad \overset{2}{K}{}^{(\lambda\mu}\widehat{F}^{\nu)}{}_{\alpha} = 0\,.$$

There exists a Killing–Maxwell 2–vector field satisfying these conditions? □

## 3. Comparison

**Gravitational structures:** If we compare infinitesimal symmetries of the kinetic energy function on the cotangent bundle with respect to the canonical symplectic 2–form and infinitesimal symmetries of the gravitational contact phase structure of the phase space of a test particle we see that they are given by Killing multi–vector fields. Killing multi–vector fields admit on $T^*\boldsymbol{E}$ functions constant of motion which generates infinitesimal symmetries of the kinetic energy function. On the other hand Killing multi–vector fields admit on $\mathcal{J}_1\boldsymbol{E}$ conserved functions which generate by the Jacobi–Hamilton lifts infinitesimal symmetries of the gravitational contact structure on $\mathcal{J}_1\boldsymbol{E}$.

**Joined structures:** If we consider an electromagnetic field, then in both approaches we get the same results for projectable infinitesimal symmetries, see Corollaries 1.1 and 2.14. On the other hand, Killing–Maxwell multi–vector fields of rank $\geq 2$ admits hidden infinitesimal symmetries on $T^*\boldsymbol{E}$. On $\mathcal{J}_1\boldsymbol{E}$ Killing–Maxwell multi–vector fields admit functions conserved with respect to the Reeb vector field of the joined structure but to obtain infinitesimal symmetries of the joined almost-cosymplectic-contact structure we need a further strong condition (2.18).

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