Uniqueness Results on Hypersurfaces with Prescribed Angle Function in Product Spaces

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Abstract

We deal with two-sided complete hypersurfaces immersed in a Riemannian product space, whose base is supposed to have sectional curvature bounded from below. In this setting, we obtain sufficient conditions which assure that such a hypersurface is a slice of the ambient space, provided that its angle function has some suitable behavior. Furthermore, we establish a natural relation between our results and the classical problem of to describe the geometry of a hypersurface immersed in the Euclidean space through the behavior of its Gauss map.

Keywords: Mean Curvature, Riemannian Products Spaces, CMC Hypersurfaces

1 Introduction

In this poster we show results of uniqueness of complete hypersurfaces immersed in a Riemannian product space $\mathbb{R}\times M^n$, where M^n is a connected, n-dimensional oriented Riemannian manifold.

When the ambient space is a Riemannian product $\overline{M}^{n+1}=\mathbb{R}\times M^n$, as it was already observed by Espinar and Rosenberg in [3], the condition that the image of the Gauss map is contained in a closed hemisphere, becomes that the angle function $\eta=\langle N,\partial_t\rangle$ does not change sign. Here, N denotes a unit normal vector field along a hypersurface $\psi:\Sigma^n\to\overline{M}^{n+1}$ and ∂_t stands for the unitary vector field which determines on $\overline{M}^{n+1}$ a codimension one foliation by totally geodesic slices $\{t\}\times M^n$.

In a recent paper [5], Rosenberg, Schulze and Spruck showed that an entire minimal graph with nonnegative height function in a product space $\mathbb{R}\times M^n$, whose base M^n is a complete Riemannian manifold having nonnegative Ricci curvature and with sectional curvature bounded from below, must be a slice. Consequently, from Theorem 2 we obtain the following:

In [1], the first author extended the technique developed by Yau in [6] in order to investigate the rigidity of entire vertical graphs in a Riemannian product space $\mathbb{R}\times M^n$, whose base M^n is supposed to have Ricci curvature with strict sign. Under a suitable restriction on the norm of the gradient of the function u which determines such a graph $\Sigma^n(u)$, he proved that $\Sigma^n(u)$ must be a slice $\{t\}\times M^n$.

2 Some Preliminaries

Let $\mathbb{R}\times M^n$ be endowed with the Riemannian metric

$$\langle,\rangle=\pi_{\mathbb{R}}^*(dt^2)+\pi_M^*(\langle,\rangle_M),$$

We say that $M^n_{t_0}=\{t_0\}\times M^n$ is a *slice* of $\overline{M}^{n+1}$.

Here $A:\mathfrak{X}(\Sigma)\to\mathfrak{X}(\Sigma)$ will denote the shape operator of Σ^n with respect to the future-pointing ($\langle N,\partial_t\rangle$) Gauss map N.

Consider now the functions attached to a hypersurface Σ^n immersed into $\mathbb{R}\times M^n$, the (vertical) height function $h=(\pi_{\mathbb{R}})|_{\Sigma}$ and the support function $\langle N,\partial_t\rangle$.

We have the gradient of h on Σ^n is

$$\nabla h=(\overline{\nabla}\pi_{\mathbb{R}})^{\top}=\partial_t^{\top}=\partial_t-\langle N,\partial_t\rangle N, \qquad (1)$$

Thus, we get

$$|\nabla h|^2=1-\langle N,\partial_t\rangle^2, \qquad (2)$$

We also have that Laplacian on Σ^n of the height function is given by

$$\Delta h=nH\langle N,\partial_t\rangle, \qquad (3)$$

where $H=\frac{1}{n}\mathrm{trace}(A)$ is the mean curvature of Σ^n relative to N.

Lemma 1 *Let $\psi:\Sigma^n\to\mathbb{R}\times M^n$ be a hypersurface with Gauss map N. Suppose that the mean curvature H is constant. Then,*

$$\Delta\langle N,\partial_t\rangle=-\left(\mathrm{Ric}_M(N^*,N^*)+|A|^2\right)\langle N,\partial_t\rangle,$$

where Ric_M is the Ricci curvature of the fiber M^n, $N^*=N-\langle N,\partial_t\rangle\partial_t$ is the projection of N onto M^n and $|A|$ is the Hilbert-Schmidt norm of the shape operator A of Σ^n.

One of the most powerful tools is the following lemma known as *generalized maximum principle* due to H. Omori and S.T. Yau [8,9].

Lemma 2 *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and $u:\Sigma^n\to\mathbb{R}$ be a smooth function which is bounded from below on Σ^n. Then there is a sequence of points $p_k\in\Sigma^n$ such that*

$$\lim_k u(p_k)=\inf u,\quad \lim_k|\nabla u(p_k)|=0\quad\text{and}\quad\lim_k\Delta u(p_k)\geq 0.$$

3 Main Results

The following results can be found in [7]. .

Theorem 1 *Let $\overline{M}^{n+1}=\mathbb{R}\times M^n$ be a Riemannian product space whose base M^n has sectional curvature K_M such that $K_M\geq-\kappa$ for some $\kappa>0$, and let $\psi:\Sigma^n\to\overline{M}^{n+1}$ be a two-sided complete hypersurface with constant mean curvature H and H_2 bounded from below. Suppose that the angle function η of Σ^n is bounded away from zero and that its height function h satisfies one of the following conditions:*

$$|\nabla h|^2\leq\frac{\alpha}{(n-1)\kappa}|A|^2, \qquad (4)$$

for some constant $0<\alpha<1$; or

$$|\nabla h|^2\leq\frac{n}{(n-1)\kappa}H^2. \qquad (5)$$

Then, Σ^n is a slice of $\overline{M}^{n+1}$.

According to Example 1 we cannot extend estimate (4) to the limit case $\alpha=1$.

When the mean curvature H is not constant, but it is does not change sign along the hypersurface we have the next.

Theorem 2 *Let $\overline{M}^{n+1}=\mathbb{R}\times M^n$ be a Riemannian product space whose base M^n has sectional curvature bounded from below, and let $\psi:\Sigma^n\to\overline{M}^{n+1}$ be a two-sided complete hypersurface which lies between two slices of $\overline{M}^{n+1}$. Suppose that the angle function η of Σ^n is not adhere to 1 or −1. If H_2 is bounded from below and H is bounded and it does not change sign on Σ^n, then $\inf_\Sigma|H|=0$. In particular, if H is constant, then Σ^n is minimal.*

From Osserman's result [4] before described, we can apply Theorem 2 we get the following:

Corollary 1 *The only two-sided complete constant mean curvature surfaces of $\mathbb{R}^3$ with Gaussian curvature bounded from below, lying between two planes and whose Gauss map is not adhere to both poles of $\mathbb{S}^2$ which are orthogonal to such planes, are planes of $\mathbb{R}^3$.*

Corollary 2 *Let M^n be a complete Riemmanian manifold with nonnegative Ricci curvature and whose sectional curvature is bounded from below. Let $\Sigma^n(u)=\{(u(x),x):x\in M^n\}\subset\mathbb{R}\times M^n$ be an entire graph of a nonnegative smooth function $u:M^n\to\mathbb{R}$, with H constant and H_2 bounded from below. If u is bounded, then $u\equiv t_0$ for some $t_0\in\mathbb{R}$.*

Furthermore, taking into account once more Theorem 2 jointly with Theorem 1.2 of [5], we also have:

Corollary 3 *Let M^n be a parabolic complete Riemmanian manifold whose sectional curvature is bounded. Let $\Sigma^n(u)=\{(u(x),x):x\in M^n\}\subset\mathbb{R}\times M^n$ be an entire graph of a smooth function $u:M^n\to\mathbb{R}$, with H constant and H_2 bounded from below. If u is bounded, then $u\equiv t_0$ for some $t_0\in\mathbb{R}$.*

Example 1 *Consider the graph of the smooth function $u:\mathbb{H}^2\to\mathbb{R}$, given by $u(x,y)=a\ln y$, in the model of half-space. Notice that $Du(x,y)=(0,ay)$ and, hence, $|Du(x,y)|^2=|a|^2$. If we take $0<|a|<1$, we have that $\Sigma^2(u)$ will be a complete spacelike surface in $-\mathbb{R}\times\mathbb{H}^2$. Its height function h satisfies*

$$|\nabla h|^2=\frac{|Du|^2}{1+|Du|^2}=\frac{|a|^2}{1+|a|^2}.$$

Consequently,

$$\langle N,\partial_t\rangle=\frac{1}{\sqrt{1+|a|^2}}.$$

The mean curvature H of $\Sigma^2(u)$ is given by

$$nH=\mathrm{Div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right),$$

where Div is the divergent on $\mathbb{H}^2$. So, using that $\mathrm{Div}=\mathrm{Div}_0-\frac{2}{y}dy$, where Div_0 denotes the divergent on $\mathbb{R}^2$, we get

$$2Hr^3=r^2y^2\Delta_0u-y^3(yQ(u)+u_y|D_0u|_0^2), \qquad (6)$$

where $r=\sqrt{1+|Du|^2}=\sqrt{1+a^2}$, Δ_0, D_0 and $|.|_0$ are the Laplacian, the gradient and the norm in the Euclidian metric, and $Q(u)=u_x^2u_{xx}+2u_xu_yu_{xy}+u_y^2u_{yy}$. Replacing $u(x,y)=a\ln y$ in equation (6), we obtain

$$H=\frac{a}{2\sqrt{1+a^2}}$$

and, since $\langle N,\partial_t\rangle$ is constant, from Lemma 1 we get

$$0=\Delta\langle N,\partial_t\rangle=-(|A|^2-|\nabla h|^2)\langle N,\partial_t\rangle.$$

Consequently,

$$|\nabla h|^2=|A|^2.$$

Furthermore, Using the fact that $|A|^2=n^2H^2-n(n-1)H_2$ we see that $H_2=0$ on $\Sigma^2(u)$. But, $H_2=\kappa_1\kappa_2$, where κ_1,κ_2 denote the eigenvalues of A. Therefore, considering $\kappa_2=0$ and using that $H=\frac{\kappa_1+\kappa_2}{2}=\frac{\kappa_1}{2}$, we obtain that $\kappa_1=\frac{a}{\sqrt{1+a^2}}$.

In this example we got a complete spacelike surface immersed in $-\mathbb{R}\times\mathbb{H}^n$ with constant mean curvature and H_2 bounded from below (it is zero).

But it is nor a slice neither maximal proving that are non trivial surfaces with constant mean curvature.

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References

[1] H.F. de Lima, *Entire vertical graphs in Riemannian product spaces*, to appear at Quaestiones Mathematicae.

[2] H.F. de Lima and U.L. Parente, *A Bernstein type theorem in $\mathbb{R}\times\mathbb{H}^n$*, Bull. Brazilian Math. Soc. **43** (2012), 17–26.

[3] J.M. Espinar and H. Rosenberg, *Complete constant mean curvature surfaces and Bernstein type theorems in $M^2\times\mathbb{R}$*, J. Diff. Geom. **82** (2009), 611–628.

[4] R. Osserman, *Proof of a conjecture of Nirenberg*, Comm. Pure Appl. Math. **12** (1959), 229–232.

[5] H. Rosenberg, F. Schulze and J. Spruck, *The Half-Space Property and Entire Positive Minimal Graphs in $M\times\mathbb{R}$*, J. Diff. Geom. **95** (2013), 321–336.

[6] S.T. Yau, *Some Function-Theoretic Properties of Complete Riemannian Manifolds and their Applications to Geometry*, Indiana Univ. Math. J. **25** (1976), 659–670.

[7] De Lima, H.F.,Lima Jr, E.A.,Parente, U. L., *Hypersurfaces with prescribed angle function*, To appear in Pac. J. of Math. .

[8] Omori, H., *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan.**19** 205-214,(1967) .

[9] Yau, S.T., *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** 201-228, (1975) .