Fock Quantization of Field Systems Coupled to Point-Particles

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Abstract

We study the Fock quantization of a system consisting of both point particle and field degrees of freedom. In order to do this, we study the details of the Hamiltonian description for the model by using the geometric constraint algorithm of Gotay, Nester and Hinds [3].

Consider an elastic string of natural length L coupled to two point masses M and m₁ located at the ends and attached to springs of zero rest length. The string and the springs are subject to restoring forces proportional to the deviations from their equilibrium.

The precise construction of the Fock space shows that the Hilbert space H of the system is not of the form H = H₉,₀ ⊗ Hₐ, which would account for a clean separation between quantum boundary and bulk degrees of freedom.

Classical Treatment

The Lagrangian of this system can be written as:
\[ L(Q,V) = \frac{1}{2} V(V) - E \left[ \nabla \cdot Q, \nabla \cdot Q \right] - m \omega^2 Q + \sum_{k=1}^{\infty} \left( \frac{M}{2} \omega^2 (V_j^2) - k \cdot k_j \right) \]

• It is easy to check that \( L(Q,V) \) is a Longitudinal mass density of the string. E Young moduli

\[ n^2 > 0 \] Spring constant per unit length of the restoring force. k, Elastic constants of the springs

Consider solutions of the form \( u(x,t) = T(x) \), we obtain:
\[ \int \Psi^* \left( u - \omega \gamma_1 \right) T \int \Psi \left( u \right) \]

Solving the PDE’s

The solution for X can be determined explicitly but it is not a Sturm-Liouville problem as X appears on the boundary conditions.

Proposition

There are infinitely many eigenvalues \( \{ \lambda_n \} \), associated with the eigenvalues \( \{ \alpha_n \} \), where

\[ \lambda_n = \mu_n^2 + \kappa_n^2 \]

\[ \kappa_n \]

\[ \mu_n \]

These types of systems share some features with field theories defined in bounded spatial regions such as Maxwell-Chern-Simons models or Yang-Mills theories.

Laplacian Operator

For a given function \( f : [0,1] \rightarrow \mathbb{R} \) with \( f \) continuous, we define the (signed) measure \( \nu \) as the one such that \( \nu([-0,1]) = \int_0^1 f(y) \, dy \) for every \( 0 \leq x \leq y \leq 1 \). It is said to be \( \mu \)-a.e. if \( \nu \mu = 0 \). Thus, if \( \nu \mu = 0 \), there exists a unique \( \mu \)-measurable function \( \theta \) in \( L^\mu (\mu) \) called Radon-Nikodym derivative, such that for every set \( A \in \mathcal{B}(0,1) \):

\[ \nu(A) = \int_A \theta \, d\mu \]

We introduce a generalized Laplace operator, defined in terms of the Radon-Nikodym derivative:

\[ \partial \nu(\mathcal{B}) = \partial \mathcal{B}(1) \]

Consider \( \lambda \), then \( \theta \) is symmetric over \( \lambda \).

Hence requiring that \( \kappa_n (1 - r_\lambda - r_\lambda) = \mu \), we recover the original problem.

Expressing our problem in this new Measure Space

Let us consider the Lagrangian \( \mathcal{L} : A \times L^0([0,1]) \rightarrow \mathbb{R} \) given by:

\[ \mathcal{L}(Q,V) = \frac{1}{2} \sum_{j=1}^n \left( \frac{dQ^j}{dt} \right)^2 + \sum_{j=1}^n \frac{1}{2} \lambda_j Q^j \]

Computing the “first variation” of the Lagrangian we get the equations of motion:

\[ u = \frac{d}{dt} \phi_0 \]

They represent the 1+1 dimensional KG equation on the interval \([0,1]\) subject to Robin boundary conditions \( u(\lambda) \) written in terms of the RN derivative and we recover indeed the same dynamics.

GNH Algorithm

After two steps, the GNH algorithm stops and we obtain the (generalized) submanifold where the dynamic takes place \( M = \lambda \times X \), and the Hamiltonian vector field on \( M \), \( M \rightarrow M \), given by:

\[ X(Q,P) = \sigma \quad \text{where} \quad X(Q,P) = \sigma (\mathbb{D} = \mathbb{D}^\lambda Q) \]

Finally we consider the completion of this space with the given norm. If such closure could be factored as \( C \times X \) then the function \( f \) would belong to it, but its expansion in the (orthonormal) Hilbert basis \( Y = X(\mathbb{D} = \mathbb{D}^\lambda Q) \) turns out to be divergent preventing such factorization and in particular the lift to the full Fock space is not a tensor product.

This construction of the Fock space is customarily used in the context of QFT in curves spacetimes but here the generalised Laplacian is defined through a measure with singular contributions of the boundaries instead of built with the help of a cut metric.

Conclusions

• Combining different types of degrees of freedom poses some interesting and non-trivial problems related to the proper characterization of the physical degrees of freedom both at the classical and quantum level.

• It is essential to build elliptic, self-adjoint operators (generalized Laplacians) with the right spectrum and associated eigenfunctions that can be mapped to the ones appearing in the resolution of the eigenvalue problem that determines the normal modes of the system.

\[ (\alpha_1, \alpha_2, \ldots, \alpha_n) \]

\[ \alpha_\text{min} \]

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