Non-commutative Quantum Gravity

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September 2014

Translating ideas from quantum gravity into C-algebraic language.*

1. What to look for:

- **Quantisation** of spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ from **categorification**: using ideas from Rieffel and from Crane.
- A spectral aspect for non-commutative geometry involving a sheaf $sh_{\mathcal{E}}$ over a site of C*-algebras together with a more tangible geometrical interpretation of the Dirac operator $\mathcal{D}$,
- An algebra invariant $\Phi_{\hookrightarrow}$ in the form of a partition function,
- Tests:
  - Classical limit theorem for $\hbar \to 0$, reproducing general relativity (Connes style) [M1].
  - Litmus test: revealing a strong consistency with the experimental fermion mass matrix [M2].

Starting point: identify the tools needed to achieve the above.

From *Fell bundles, C*-bundles, C*-categories, and C*-dynamical systems,*

the following picture emerges: "fuzzy points" hiding in the fibres, where $|\mathcal{D}|$ is interpreted as a connection on the C*-bundle.

3. Dirac operator $\mathcal{D}$

Let $\hbar = 1$, $T\mathcal{X} \to \mathcal{X} \times \mathcal{X}$. $\mathcal{D}$ is given by a self-adjoint section of $\mathcal{E}$ supported on a bisection of the pair groupoid $G = \mathcal{X} \times \mathcal{X}$ over $G_0 = \mathcal{X} \cong \mathrm{Ob}(\mathcal{C})$,

$$\mathcal{D}_{\text{finite}} = \begin{pmatrix} 0 & M^* & 0 & 0 \\ M & 0 & 0 & 0 \\ 0 & 0 & 0 & M^T \\ 0 & 0 & \bar{M} & 0 \end{pmatrix}$$

where M is an element of a representable sheaf called a "tangent sheaf" $sh_{\mathcal{E}}$,

$$sh_{\mathcal{E}} : \mathrm{Hom}(-, \mathcal{A}) \to \mathcal{C}, \quad \text{for each } \mathcal{A} \in \mathrm{Ob}(\mathcal{C}).$$

Non-commutativity arises for a second reason, namely $[q_i, q_j] \neq 0$...
Configuration algebra $C^*(\mathcal{E}^0)$ generalises $C^*(\mathcal{X})$,
Observable algebra $C^*(\mathcal{E})$, generalises $C^*(\mathcal{X} \times \mathcal{X})$.

2. A "deformed" spectral triple takes this form:

A ***spectral C*-category*** is a small full C*-category $\mathcal{C}$ equipped with a continuous global self-adjoint section (or coretraction) σ of its domain map such that $r \circ \sigma : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ is bijective, where r is the range map of $\mathcal{C}$. [M1].

...and draws on Rieffel quantisation and the tangent groupoid quantisation.

☺

(A full C*-category $\mathcal{C}$ fibred over the maximal equivalence relation on its object space $\mathrm{Ob}(\mathcal{C})$ is equivalent to a saturated unital Fell bundle $(\mathcal{E}, \pi, G)$ over a pair groupoid $G = \mathcal{X} \times \mathcal{X}$ with $\mathcal{X} \cong \mathrm{Ob}(\mathcal{C})$. The fibres of $\mathcal{E}$ are Morita equivalence bimodules [BCL1].)

☼

A category $\mathfrak{C}_{MR}$ of unital C*-algebras $\mathcal{A}$, $\mathcal{B}$ and Morita-Rieffel equivalence $\mathcal{A}$-$\mathcal{B}$-bimodules $\mathcal{M}_{\mathcal{AB}}$ has a rich geometrical significance.

$$\mathcal{M}_{\mathcal{AB}} \otimes \mathcal{M}^*_{\mathcal{BA}} \cong \mathcal{A}$$
$$\mathcal{M}^*_{\mathcal{BA}} \otimes \mathcal{M}_{\mathcal{AB}} \cong \mathcal{B}$$

$$\delta_{\mathcal{D}} : \mathcal{A} \oplus \mathcal{B} \to \Omega^1_{\mathcal{D}}(\mathcal{A} \oplus \mathcal{B})$$
$$\delta_{\mathcal{D}} : a \mapsto [\mathcal{D}, a].$$

4. Algebra is a laboratory

1. Non-commutative TQFT,

   $$\mathcal{Z} : \mathcal{C} \to \mathrm{Hilb}$$

   (Since $\mathcal{C}$ is a concrete C*-category, $\mathcal{Z}$ will always exist as a representation, or *-functor of C*-categories.)

   The category $\mathcal{C}$ is even more like Hilb than nCob is.

2. 1-parameter groups of inner automorphisms of C*-bundles, ⟲

A ***reversible C*-bundle dynamical system*** $(\mathcal{E}^0, \mathcal{G}_\sigma)$ is given by a C*-bundle $(\mathcal{E}^0, \pi, \mathcal{X})$ and a 1-parameter covariance group $\mathcal{G}_\sigma$ of C*-bundle inner automorphisms. (Covariance means $\pi \circ g = f_0 \circ \pi$ where $f_0 \in \mathrm{Diff}(\mathcal{X})$).

...leading to an **algebra invariant:**
(equivalent to a groupoid 2-cocycle $\omega : G \times G \to \mathbb{T}$),

$$\Phi_{\hookrightarrow} = \sum_{m=1}^{n} \prod_{i=1}^{m} U_{g_i}, \quad i = 1, ..., m, \quad m = 1, ..., n.$$

(Work in progress with Bertozzini, Conti, Resende... See also modular quantum gravity [BCL2], and [BCL3,4,5] spectral theory of C*-categories.)