Curvature identities and Gauss-Bonnet type theorems

Navarro, A. & Navarro J.

ICMat, CSIC, Spain; Departamento de Matemáticas, UEx, Spain

a.navarro.garmendia@icmat.es; navarrogarmendia@unex.es

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GOBIERNO DE EXTREMADURA

1. Abstract

For a fixed $n \in \mathbb{N}$, the curvature tensor of a pseudo-Riemannian metric, as well as its covariant derivatives, satisfy certain identities that hold on any manifold of dimension less or equal than n. As an example, on any pseudo-Riemannian manifold of dimension 2 the following relation holds:

$$Ric - \frac{r\, g}{2} = 0 \;, \tag{1}$$

where Ric denotes the Ricci tensor and r the scalar curvature of g.

In this poster, we re-elaborate recent results by Gilkey-Park-Sekigawa regarding these curvature identities on pseudo-Riemannian manifolds (see [4]). To this end, we use the classical theory of natural operations, that allows us to simplify some arguments and to generalize some results of [2], both by dropping a symmetry hypothesis and by including p-covariant curvature identities, for any even p.

In the final section, we state how to use this theorem to refine a classical result by Gilkey ([1]), regarding the uniqueness of the Gauss-Bonnet theorem.

2. Universal tensors and dimensional restriction

A **natural tensor in dimension** n**, associated to metrics of signature** (n_+, n_-), is an assignment T that, for any metric g on any smooth manifold X of dimension n, produces a tensor $T(g)$ on X, satisfying:

$$T(\tau^* g) = \tau^* T(g) \;,$$

for any local diffeomorphism $\tau\colon Y \to X$ between smooth manifolds of dimension n.

A natural tensor T is **homogeneous of weight** $w \in \mathbb{R}$ if, for any metric g on any smooth manifold of dimension n, and any positive real number $\lambda > 0$, it holds:

$$T(\lambda^2 g) = \lambda^w\, T(g) \;.$$

Therefore, the $\mathbb{R}$-vector space of homogeneous natural tensors will be denoted:

$$\mathsf{T}_{p,w}[n] := \left[\begin{array}{c} \text{Natural } p\text{-tensors } T \text{ in dimension } n \\ \text{homogeneous of weight } w \end{array} \right] \;.$$

The **dimensional reduction** of natural tensors is the linear map:

$$\mathsf{T}_{p,w}[n] \xrightarrow{\;r_n\;} \mathsf{T}_{p,w}[n-1] \quad , \quad r_n(T)(g) := i^* \left(T(g + \mathrm{d}t^2) \right) \;.$$

where i denotes the embedding:

$$i\colon X \hookrightarrow X \times \mathbb{R} \qquad , \qquad x \;\mapsto\; (x, 0) \;.$$

These linear maps establish a projective system:

$$\ldots \xrightarrow{\;r_{n+1}\;} \mathsf{T}_{p,w}[n] \xrightarrow{\;r_n\;} \mathsf{T}_{p,w}[n-1] \xrightarrow{\;r_{n-1}\;} \ldots \;.$$

A **universal tensor**, homogeneous of weight w, is an element of the inverse limit

$$\mathsf{T}_{p,w} := \varprojlim \mathsf{T}_{p,w}[n] \;.$$

Example. *The metric g, the Riemann-Christoffel tensor R, the Ricci tensor Ricc, and the scalar curvature r are all universal tensors. For any fixed $\lambda \in \mathbb{R}$, the tensor λg is a universal tensor. However, $(\mathrm{tr}\,\mathrm{Id})\, g$, $(-1)^{\dim X} g$ or $(-1)^{n_+} g$ are not universal tensors.*

Example. *For any $k, \bar{p} \geq 0$, let us define the $(2\bar{p})$-covariant universal tensors:*

$$\mathsf{S}_{2\bar{p},k} := c\left(R^k \otimes g^{2k+\bar{p}} \right) \;.$$

where c denotes the contraction operator:

$$\left(\Lambda^{2k} TX \otimes \Lambda^{2k+\bar{p}} T^*X\right) \otimes \left(\Lambda^{2k} TX \otimes \Lambda^{2k+\bar{p}} T^*X\right) \xrightarrow{c} \Lambda^{\bar{p}} T^*X \otimes \Lambda^{\bar{p}} T^*X$$

3. Dimensional curvature identities

A **dimensional curvature identity** in dimension n is an element of the projective space associated to the vector space:

$$\mathsf{K}_{p,w}[n] := \mathrm{Ker}\left[\mathsf{T}_{p,w} \longrightarrow \mathsf{T}_{p,w}[n] \right] \;.$$

Example. *For any fixed $\bar{p}, k \geq 0$ (apart from the cases $k = 0$ and $\bar{p} = 0, 1$) the tensor $\mathsf{S}_{2\bar{p},k}$ vanishes whenever $\dim X < 2k + \bar{p}$, because the form $g^{2k+\bar{p}}$ is identically zero.*

If $T \in \mathsf{T}_{2\bar{p},w}$ is a $(2\bar{p})$-universal tensor, homogeneous of degree w, then we may write, without loss of generality ([4]):

$$w = 2(\bar{p} - k)$$

for some $k \geq 0$.

Theorem. *Consider covariant tensors with an even number $2\bar{p}$ of indices ($\bar{p} \geq 0$), associated to non-singular metrics.*
For any weight $w = 2(\bar{p} - k)$, with $k \geq 0$, the following holds:

- *If $n \geq 2k + \bar{p}$, there are no dimensional curvature identities of weight w; that is,*

  $$\mathsf{K}_{2\bar{p},w}\,[n] \;=\; 0 \quad , \quad \text{for } n \geq 2k + \bar{p} \;.$$

- *The vector space $\mathsf{K}_{2\bar{p},w}\,[2k + \bar{p} - 1]$ is generated by the universal tensors $\{\sigma \cdot \mathsf{S}_k\}_{\sigma \in S_{2\bar{p}}}$, and hence has dimension:*

  $$\dim\left(\mathsf{K}_{2\bar{p},w}\,[2k + \bar{p} - 1]\right) \;=\; \bar{p}\;(\bar{p}+1) \cdot \ldots \cdot (2\bar{p}-2)(2\bar{p}-1) \;,$$

  or reduces to a single identity, when $\bar{p} = 0$.

4. Gauss-Bonnet type theorems

The aforementioned scalar differential invariants $\mathsf{S}_{0,k}(g)$ appear inside the well-known Gauss-Bonnet theorem:

Theorem (Gauss-Bonnet)**.** *Let X be a compact, orientable smooth manifold of even dimension $n = 2k$.*
For any Riemannian metric g on X, it holds:

$$\chi(X) \;=\; \frac{1}{(8\pi)^k k!} \int_X \mathsf{S}_{0,k}(g) \mathrm{Vol}_g \;.$$

In [4], we plan to use the Theorem stated in Section 3 to prove the following result (see also [1]):

Theorem. *Let $\mathcal{L}$ be a universal, homogeneous (scalar) differential invariant.*
Assume there exists a compact, orientable smooth manifold X of dimension n such that:

$$\int_X \mathcal{L}(g)\mathrm{Vol}_g \text{ is a non-zero number, independent of the metric } g \text{ on } X.$$

Then $n = 2k$ is even, and there exist $\mu \in \mathbb{R}$ and a natural vector field D such that:

$$\mathcal{L} = \mu\, \mathsf{S}_{0,k} + \mathrm{div}\, D \;.$$

References

[1] P. Gilkey: *Local invariants of an embedded Riemannian manifold*, Ann. Math. (1975).

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[3] P. Gilkey, J. H. Park, K. Sekigawa: *Universal curvature identities and Euler Lagrange Formulas for Kaehler manifolds*, ArXiv: 1311.2622.

[4] A. Navarro, J. Navarro: *Dimensional curvature identities on pseudo-Riemannian geometry*, ArXiv: 1310.2878.