# Intelligent spin states constructed from $SU_q(2)$ coherent states

M. Reboiro [1] and O. Civitarese [1]

[1] *Dept of Physics, University of La Plata c.c. 67 1900, La Plata, Argentina*

## Abstract

It is shown that $SU_q(2)$ Coherent Spin States behave as Intelligent Spin States on two orthogonal components which are perpendicular to the direction of the mean value of the spin operator.

## Introduction

The study of spin squeezed atoms and ions play a main role in high precision measurements [1, 2]. Spin squeezed states have been detected in recent experiments [3, 4, 5]. The authors of [[4]] have reported the observation of coherent squeezed states in a Bose-Einstein condensate of rubidium atoms ($^{87}$Ru) and presented a direct experimental demonstration of interferometric phase precision beyond the standard quantum limit in a nonlinear Ramsey interferometer. Intelligent Spin States (ISS) are states which minimize Heisenberg Uncertainty Relations. The notion of ISS have been introduced by C. Argone and co-workers in Ref. [[6]]. Following [6], a considerable amount of work was devoted to the study of both the properties of ISS [7] as well as to the construction of such states [8]. As an example, the authors of [8] have constructed ISS as the steady solutions of a collection of two-level atoms interacting with a broadband squeezed radiation field (see also Ref. [[9]] ) The search for minimum uncertainty relations and ISS is a matter of interest for quantum groups and Hopf algebras [10]. Quantum groups have been successfully applied to the study of different physical problems. The interaction between spin-states and a radiation field has been described in terms of an effective q-deformed Hamiltonian with spin-spin interactions, both for two and three atomic level systems [11, 12, 13]. Here, we shall show that $su_q(2)$ Coherent Spin States behave as Intelligent Spin States and that they can be used to modelled squeezed Dicke states [8]. In doing so, we shall follow the work of Ref. [[14]] and study angular momentum $su_q(2)$ uncertainty relations, for a q-deformed coherent spin state.

## Formalism

As demonstrated by Agarwall and co-workers [8], Intelligent Spin States can be generated when a collection of two-level atoms is exposed to a squeezed radiation bath. The authors of [8] have studied the properties of a system of N identical two-level atoms of frequency $\omega_0$ interacting with broadband squeezed radiation field. The Hamiltonian of the system reads

$$H = \omega_0 J_z + \int \mathrm{d}\omega\ \omega\ a^\dagger(\omega)a(\omega) + \int \mathrm{d}\omega\ g(\omega)(a^\dagger(\omega)J_- + J_+ a(\omega)). \quad (1)$$

The operators $J_\pm$, $J_z$ are the angular momentum operators corresponding to the spin value $N/2$. The atom-field interaction is given by $g(\omega)$. The annihilation and creation operators $a(\omega)$ and $a^\dagger(\omega)$ satisfy the usual commutation relations $\left[a(\omega_1), a^\dagger(\omega_2)\right] = \delta(\omega_1 - \omega_2)$. The field operators are acting on the squeezed vacuum state.The reduced density matrix of the atomic system can be derived by using standard master equation methods. In [[8]], G. S. Agarwall and R. R. Puri have proved that the steady-state solution of the problem is $\rho = |\Phi_0><\Phi_0|$, with

$$|\Phi_0>= A_0 \exp^{\theta J_z} \exp^{-i\frac{\Pi}{2}\theta J_y}|0>, \quad (2)$$

where $|0>$ is eigenstate of $S_z$ with eigenvalue $m = 0$ and $A_0$ is a normalization constant. The expectation values and directions, of the components of the angular momentum $J_i$ on $|\Phi_0>$ obey the relations

$$\Delta^2 J_x \Delta^2 J_y = |<J_z>|/2. \quad (3)$$

Thus the steady state $|\Phi_0>$ is an intelligent spin state, i.e.: (i) it is an eigenstate of the operator $\cosh(|\xi|)J_- + \sinh(|\xi|)J_+$ with zero eigenvalue, (ii) the component $J_x$ is squeezed ($\zeta_x^2 = 2\Delta^2 J_x/|<J_z>|<1$) at the expense of the $J_y$ component($\zeta_y^2 = 2\Delta^2 J_y/|<J_z>|>1$). In this analysis we have taken $\zeta$ as the squeezing parameter of the radiation field, thus $\zeta_x^2 = \exp -|\zeta|$ and $\zeta_y^2 = \exp|\zeta|$.(see [9] for the experimental consequences of this definition) Now, we shall probe that the state (2) can be constructed from a $su_q(2)$ coherent state. Let us briefly review the basis notion about the $su_q(2)$ algebra. The generators of the $su_q(2)$ algebra, $S_+, S_-, S_z$, obey the relations

$$[S_z, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = [2S_z]_q, \quad (4)$$

with

$$[x]_q = \frac{\sin(zx)}{\sin(z)}. \quad (5)$$

where we have adopted $q = e^{iz}$. As in the case of the $su(2)$ algebra, the irreducible representation of the $su_q(2)$ are labelled by $S$. We shall denote the orthonormal basis of representation as $|SM>$, with $M = -S, -S+1, ..., S$. The action of the generators of the $su_q(2)$ on this basis obeys ($\hbar = 1$)

$$\begin{aligned} S_z|S\ M> &= M\ |S\ M>, \\ S_\pm|S\ M> &= \sqrt{[S \mp M]_q[S \pm M + 1]_q}\ |S\ M \pm 1>. \end{aligned} \quad (6)$$

For a given S-representation of the $su_q(2)$ algebra, q-deformed CSS states, $|\eta>$, are defined by

$$\begin{aligned} |\eta\rangle &= \mathcal{N} e_q^{\eta S_+}\ |\ S\ -S> \\ &= \mathcal{N} \sum_{k=0}^{2S} \eta^k \begin{bmatrix} 2S \\ k \end{bmatrix}_q^{1/2} |S-S+k\rangle, \end{aligned} \quad (7)$$

where $e_q$ is the q-exponential function

$$e_q^x = \sum_k \frac{x^k}{[k]_q!}, \quad (8)$$

and

$$\begin{bmatrix} 2\ S \\ k \end{bmatrix}_q = \frac{[2\ S]_q!}{[2\ S-k]_q![2\ S]_q!}, \quad (9)$$

is the q-binomial coefficient [10]. The normalization constant $\mathcal{N}$ is written

$$\mathcal{N} = \frac{1}{\sqrt{(1+|\eta|^2)_q^{2S}}}, \quad (10)$$

with the usual q-binomial expansion

$$(1+|\eta|^2)_q^{2S} = \sum_{k=0}^{2S} |\ \eta\ |^{2\ k} \begin{bmatrix} 2S \\ k \end{bmatrix}_q. \quad (11)$$

The components $x$ and $y$ of the spin can be written, as usual, in terms the ladder operators $S_+$ and $S_-$:

$$\begin{aligned} S_x &= \frac{S_+ + S_-}{2}, \\ S_y &= \frac{S_+ - S_-}{2\mathrm{i}}. \end{aligned} \quad (12)$$

The uncertainty relation respect to the state (7) reads

$$\Delta^2 S_x \Delta^2 S_y \geq \frac{1}{4}|<[S_x, S_y]>|^2. \quad (13)$$

Following the procedure of M. Kitagawa and M. Ueda [14], we shall look for the mean value of the spin operator, $<\mathbf{S}>$, for the state of Eq.(7). We shall define a new system of axes, such that the $z'$ direction coincides with the direction of the mean value of the total spin $<\mathbf{S}>$; that is the direction determined by the unitary vector $\{\breve{n}_{x'}, \breve{n}_{y'}, \breve{n}_{z'}\}$, with

$$\breve{n}_{z'} = \frac{<\mathbf{S}>}{|<\mathbf{S}>|}. \quad (14)$$

In this new system

$$\Delta^2 S_{x'} \Delta^2 S_{y'} = \frac{1}{4}|<[S_{x'}, S_{y'}]>|^2. \quad (15)$$

For $z = 0$, the usual $su(2)$ algebra is recovered. In this limit, the coherent spin state satisfies the minimum uncertainty relationship, with uncertainties equally distributed over any two orthogonal components normal to the vector defined by $<\mathbf{S}>$. For $z \neq 0$, $\Delta^2 S'_x$ and $\Delta^2 S'_y$ are, in general, not equal, thus the q-coherent spin state resembles the state of a correlated system. In view of the previous results we shall adopt, as indicators of the relative fluctuations, the quantities

$$\begin{aligned} \zeta_{x'}^2 &= \frac{2\Delta^2 S_{x'}}{|<[S_{x'}, S_{y'}]>|}, \\ \zeta_{y'}^2 &= \frac{2\Delta^2 S_{y'}}{|<[S_{x'}, S_{y'}]>|}. \end{aligned} \quad (16)$$

The deformation parameter $z$, of Eq. (5), can be related to the value of the factor of the field state, $\xi$, by the comparison of the parameter $\zeta_x$ for the ISS and $\zeta_{x'q}$ of the q-deformed CSS. As an example with analytical solutions, we shall take the case of a system with $S = 1$, and for $\phi_0 = \pi$. It is straightforward to compute the parameters of Eq. (16), they read

$$\begin{aligned} \zeta_{x'q}^2 &= \sqrt{\cos^2(\theta_0) + \cos^2(z)\sin^2(\theta_0)}, \\ \zeta_{y'q}^2 &= \frac{1}{\sqrt{\cos^2(\theta_0) + \cos^2(z)\sin^2(\theta_0)}}. \end{aligned} \quad (17)$$

Thus, in order to model the steady solution of the Hamiltonian of Eq. (2) for $N = 2$, the parameter $z$ should be taken as $z = \arcsin\left(\sqrt{(1 - e^{-2\xi})\sin^{-2}(\theta_0)}\right)$. The definitions of Eq.(16) are consistent with the property (15). However, due to the fact that in constructing the new system of orthogonal axes we have not performed an $SU_q(2)$-rotational transformation, the commutation relations between the rotated components of the spin do not coincide with the commutation relations valid in the original system where the inequality (13) holds.

## Results and Discussion

In this section we shall present some numerical results to test the validity of our conjecture. We shall take an arbitrary large value of the spin, in order to get larger values of the spin components and their fluctuations. Figure 1 shows the values of the parameters of Eq. (16), as a function of the deformation. We have chosen a q-deformed CSS with total spin $S = 10$. The orientation angles of the state are $\theta_0 = \pi/3$ and $\phi_0 = 0$. These values are arbitrary ones, a choice which is not affecting the validity of our conclusions, as we shall see later on. As it can be seen from Figure 1, as $\zeta_{y'q}^2$ increases $\zeta_{x'q}^2$ decreases, while the product $\zeta_{y'q}^2\ \zeta_{x'q}^2$ is constant and it equals unity. Thus, the q-deformed CSS behaves as an intelligent spin state. Due to the properties of the q-deformed coherent $|\eta>$ of Eq. (7), it can be used as the stationary solution of the Hamiltonian of Eq. (2). The correspondence between the value $\xi$, and the value of $z$ of the q-deformed CSS, for the state of Figure 1, is shown in Figure 2. As seen from the Figure, the parameter $z$ is a smooth function of $|\xi|$. In order to show that the features displayed in Figure 1 remain, regardless the particular choice of the orientation angle $\theta_0$, we have calculated the parameters (16) for a set of values of $\theta_0$. Indeed, the curves displayed in Figure 3 show that the product $\zeta_{y'q}^2\ \zeta_{x'q}^2$ equals unity, as it is the case of the results shown in Figure 1. The present result may indicate that q-deformed coherent spin states can be regarded as intelligent spin states, when the orientation of the components of the spin are properly defined. The comparison of the mean values of the operators $<J_z>$ (ISS) and $<S_z>$ (q-deformed CSS), also, supports this idea [15]. Because of the fact that q-deformed coherent spin states can be used to model real squeezed states, they should be of interest in spectroscopy [1, 2]. In a recent experiment [4], collisions in ultracold atomic gases have been used to induce quadrature spin squeezing in two-component Bose condensates. The analysis of [4] states that a nonlinear atom interferometer surpasses classical precision limit. These type of states have potential application to mesoscopic systems, as for instance molecular spin clusters which are paradigmatic cases to study the cross over between quantum and classical behavior

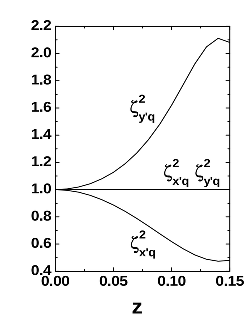

Fig. 1: Parameters, $\zeta_{y'q}^2$ and $\zeta_{x'q}^2$ of Eq. (16), as a function of the deformation parameter, $z$. The figure shows the results obtained for a q-coherent spin state. For this calculation we have adopted the values $S = 10$, $\theta_0 = \pi/3$ and $\phi_0 = 0$.

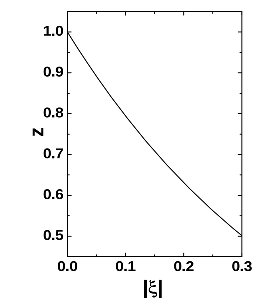

Fig. 2: Corresponding deformation parameter, $z$, of the CSS $|\eta\rangle$ of Eq. (7), for different values of the vacuum squeezing field, $\xi$. The figure shows the results obtained for a q-coherent spin state. For this calculation we have adopted the values $S = 10$, $\theta_0 = \pi/3$ and $\phi_0 = 0$.

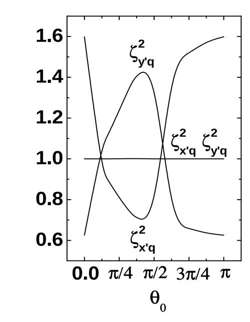

Fig. 3: $\zeta_i^2$, as a function of the orientation angle $\theta_0$ of a initial q-coherent spin-state. We have considered a system with total spin $S = 10$, and an initial state with $z = 0.08$ and $\phi_0 = 0$.

## Conclusions

In this work we have reported on the realization of q-deformed coherent spin states as intelligent spin states. We have proved that they can be used to modelled the stationary solutions of a system of two level atoms in a squeezed radiation bath. From the point of view of practical applications, q-deformed $su_q(2)$ coherent states should be of interest in spectroscopy, as well as in preparation of entanglement states trough a beam splitter. Work is in progress concerning formal aspects of the conjecture, as well as the application of this type of states to mesoscopic systems.

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