Real hypersurfaces with two principal curvatures in complex projective and hyperbolic planes

Cristina Vidal Castiñeira

Joint work with: J. C. Díaz Ramos, M. Domínguez Vázquez

Department of Geometry and Topology,
University of Santiago de Compostela, Spain
E-mail: cristina.vidal@usc.es

XXIII International Fall Workshop on Geometry and Physics, Granada 2014
September 2-5, 2014, Granada, Spain.

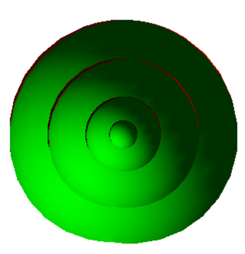
Introduction

$\bar{M}^n(c) = \mathbb{C}P^n$ or $\mathbb{C}H^n$ (constant holomorphic curvature $c \neq 0$).
J complex structure, $\bar{\nabla}$ Levi-Civita conection.
$M \subset \bar{M}^n(c)$ real hypersurface with unit normal vector ξ.
$S\xi = -\bar{\nabla}_X \xi$ shape operator.
Principal curvatures: eigenvalues of S.

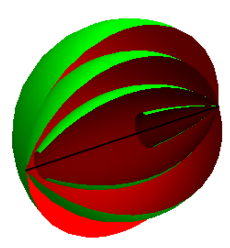
Main problem:
Classification of real hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ with a fixed number of principal curvatures.

- 1 principal curvature: impossible [7].
- 2 principal curvatures:
  - $n \geq 3$: [2] for $\mathbb{C}P^n$ and [4] for $\mathbb{C}H^n$. They are homogeneous ($\Rightarrow$ constant principal curvatures) and Hopf ($SJ\xi = \lambda J\xi$).

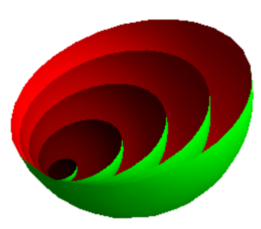
    Examples in $\mathbb{C}H^n$:



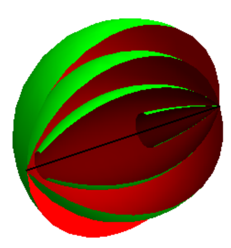
Geodesic spheres



Tubes around a totally geodesic $\mathbb{C}H^{n-1}$



Horospheres



Tubes of radius $r = \frac{1}{\sqrt{-c}}\log(2+\sqrt{3})$ around a totally geodesic $\mathbb{R}H^n$

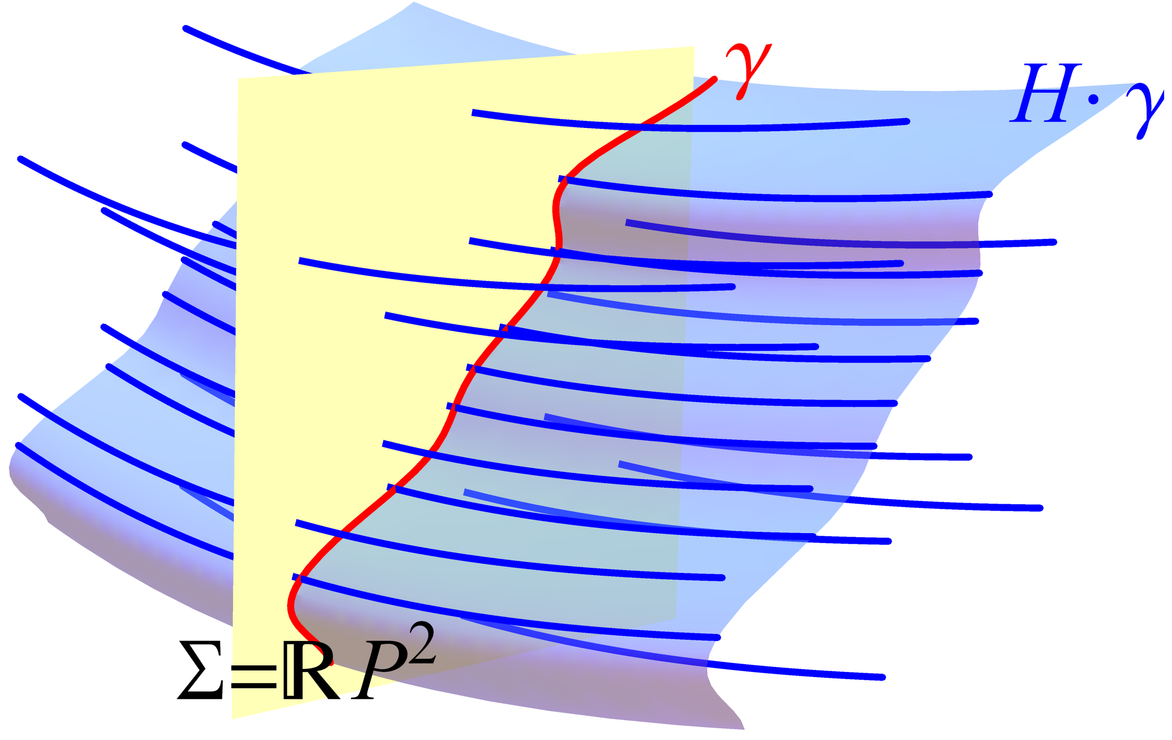
  - $n = 2$:

[5, Question 9.2]:
Are there hypersurfaces in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ that have two principal curvatures, other than the standard examples?

Construction idea

H group acting polarly on $\bar{M}^n(c)$ with cohomogenity two [1, 6] (that is, there exists a two-dimensional submanifold Σ that intersects all the orbits of H orthogonally).
$\gamma : (-\varepsilon, \varepsilon) \to \Sigma$: regular curve in Σ.
$H \cdot \gamma = \{h(\gamma(t)) : h \in H, t \in (-\varepsilon, \varepsilon)\}$
Generically $H \cdot \gamma$ has 3 principal curvatures.
$H \cdot \gamma$ has 2 principal curvatures $\Leftrightarrow$ γ satisfies an ODE.
By the existence of solutions to ODEs, such hypersurfaces exist.

Idea if $\bar{M}^2(c) = \mathbb{C}P^2$:



If $\bar{M}^2(c) = \mathbb{C}H^2$, then $\Sigma = \mathbb{R}H^2$.

Main Theorem [3]
Any hypersurface with 2 nonconstant principal curvatures in $\bar{M}^2(c)$ that is not Hopf, is locally congruent to an open part of a real hypersurface constructed as above.

References

[1] J. Berndt, J. C. Díaz-Ramos: Real hypersurfaces with constant principal curvatures in complex hyperbolic spaces, *Ann. Global Anal Geom.* **43** (2013), 99-106
[2] T. E. Cecil, P. J. Ryan: Focal sets and real hypersurfaces in complex projective space, *Trans. Amer. Math. Soc.* **269** (1982), no. 2, 481-499.
[3] J. C. Díaz-Ramos, M. Domínguez-Vázquez, C. Vidal-Castiñeira: Real hypersurfaces with two principal curvatures in complex projective and hyperbolic planes, arXiv:1310.0357v1 [math.DG].
[4] S. Montiel: Real hypersurfaces of a complex hyperbolic space, *J. Math. Soc. Japan* **37** (1985), 515-535.
[5] R. Niebergall, P. J. Ryan: Real Hypersurfaces in Complex Space Forms, *Tight and Taut Submanifolds*, MSRI Publications, Volume **32**, 1997.
[6] F. Podestà, G. Thorbergsson: Polar actions on rank-one symmetric spaces, *J. Differential Geom.* **53** (1999), no. 1, 131-175.
[7] Y. Tashiro, S. I. Tachibana: On Fubinian and C-Fubinian manifolds, *Kodai Math. Sem. Rep.* **15** (1963), 176-183.

Acknowledgements: The authors have been supported by projects GRC 2013-045 and MTM2013-41335-P with FEDER funds (Spain), and a fellowship from Plan I2C-2013 (Xunta de Galicia).