The symplectic normal space of a Lagrangian fibration

D. Rodríguez[1] & M. Teixidó-Román[2]

[1] Universidad de La Laguna, Spain. Email: drodrigd@ull.edu.es

[2] Universidad Politécnica de Cataluña, Spain. Email: mteixido@gmail.com

Abstract

A fibration $\pi : L \to Q$ is Lagrangian if the total space (L, ω) is a symplectic manifold and the fibers are Lagrangian. Assume that a Lie group G acts on L preserving the symplectic and fibered structures and it has an associated momentum map $J : L \to \mathfrak{g}^*$, which is (in general) non-equivariant. In this setting, we will give a description of the symplectic normal space at any point $z \in L$. This result is a generalization of previous results on magnetic cotangent bundles and is a first step towards the construction of semi-local normal forms for this structure.

Introduction

(M, ω, G, J) Hamiltonian G-spaces

- (M, ω) symplectic manifold
- G a Lie group with Lie algebra $\mathfrak{g}$
- $G \times M \to M$ proper and symplectic (left) action
- $J : M \to \mathfrak{g}^*$ momentum map

The Marle-Guillemin-Sternberg normal form of M at z is a local model of this Hamiltonian G-space. An important part of the computation of this local model is the description of the symplectic normal space V of the G-action on M. V is a symplectic subspace of T_zM which is given by the Witt-Artin decomposition.

Notation:

- $G_z = \{g \in G : g \cdot z = z\}$ the isotropy subgroup of $z \in M$
- $\mathfrak{g}_z$ the Lie algebra of G_z
- $\xi_M(z)$ the infinitesimal generator at z associated with $\xi \in \mathfrak{g}$
- $\mathfrak{g} \cdot z$ the set of the infinitesimal generators at z

Cotangent-lifted action

An interesting example of Hamiltonian G-space is the cotangent lift of a proper action of a Lie group G on a manifold Q.

- $M = T^*Q$
- $\omega = \Omega_Q$ canonical symplectic structure on T^*Q
- $G \times T^*Q \to T^*Q$ cotangent lift of the G-action on Q
- $J_Q : T^*Q \to \mathfrak{g}^*$ (G-equivariant) momentum map

$$\langle J_Q(\alpha_q), \xi\rangle = \alpha_q(\xi_Q(x)),\ \alpha_q \in T^*Q,\ \xi \in \mathfrak{g}$$

When Q is also a Riemannian manifold and the action of G on Q is by isometries, we have the following identification of the symplectic normal space of (T^*Q, Ω_Q) at $\alpha_q \in T^*Q$:

$$(V, \omega_V) \simeq (V_\mu \oplus T^*B, \omega_\mu(\mu) + \Omega_B)$$

- $B = ((\mathfrak{g}_q \cap \mathfrak{g}_\mu) \cdot \alpha_q)^\circ \cap (\mathfrak{g} \cdot q)^\perp \subset T_qQ$
- Ω_B canonical symplectic form of T^*B
- $(\mathcal{O}_\mu, \omega_\mu)$ the symplectic coadjoint orbit
- V_μ symplectic normal space of $(\mathcal{O}_\mu, \omega_\mu, G_q, J_\mu)$

This situation can be more general when one considers as symplectic structure on T^*Q the canonical structure modified by a magnetic term. But both cases are examples of Lagrangian fibration with a proper G-action preserving the symplectic and fibered structure and an associated (equivariant and non-equivariant, respectively) momentum map.

The symplectic normal space of the Lagrangian fibration

Hypotesis

- $\pi : (L, \omega) \to Q$ a Lagrangian fibration
- G a Lie group which acts properly on L preserving the symplectic and fibered structures
- $J : L \to \mathfrak{g}^*$ a momentum map

The Problem

Motivated by the previous examples, our aim is to give a simple description of the symplectic normal space of (L, ω, G, J) at a point $z \in L$, as a previous step to compute the Marle-Guillemin-Sternberg normal form.

Affine coadjoint action

An important tool for our purpose is the affine coadjoint action induced by the one-cocycle $\sigma : G \to \mathfrak{g}^*$ associated with J

$$\sigma(g) := J(g \cdot z) - \mathrm{Ad}^*_{g^{-1}}(J(z))$$

If L is connected, σ does not depend on the point $z \in L$. In addition, we can define a new action of G on $\mathfrak{g}^*$

$$g \cdot \mu := \mathrm{Ad}^*_{g^{-1}}\mu + \sigma(g).$$

This action is called the *affine coadjoint action* associated with σ. The orbit of this action at $\mu \in \mathfrak{g}^*$ is denoted by $\widetilde{\mathcal{O}}_\mu$. Just as the case of the coadjoint orbit, $\widetilde{\mathcal{O}}_\mu$ is a symplectic manifold with the following symplectic structure $\widetilde{\omega}_\mu$

$$\widetilde{\omega}_\mu(\nu)(\xi_{\mathfrak{g}^*}(\nu), \bar{\xi}_{\mathfrak{g}^*}(\nu)) = -\langle \nu, [\xi, \bar{\xi}]\rangle + \Sigma(\xi, \bar{\xi}),$$

where $\Sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is given by

$$\Sigma(\xi, \bar{\xi}) := \langle T_e\sigma(\xi), \bar{\xi}\rangle = J^{[\xi, \bar{\xi}]} - \{J^\xi, J^{\bar{\xi}}\},$$

and $\{\cdot, \cdot\}$ is the Poisson bracket associated with ω.

The Result

Fix $z \in L$. We denote by $q = \pi(z)$ and $\mu = J(z)$. Consider a decomposition

$$T_qQ = (\mathfrak{g} \cdot q) \oplus S.$$

In this case, the symplectic normal space of L is described by the direct sum of the cotangent bundle of the linear space B and the symplectic normal space of $(\widetilde{\mathcal{O}}_\mu, \widetilde{\omega}_\mu, G_q, \widetilde{J}_\mu)$ at μ.

$$(V, \omega_V) \simeq (\widetilde{V}_\mu \oplus T^*B, \widetilde{\omega}_\mu(\mu) + \Omega_B)$$

- $B = ((\mathfrak{g}_q \cap \widetilde{\mathfrak{g}}_\mu) \cdot z)^\circ \cap S \subset T_qQ$
- $\widetilde{V}_\mu$ symplectic normal space of the affine coadjoint orbit of μ
- Ω_B canonical symplectic form of T^*B
- $\widetilde{\omega}_\mu(\mu)$ the symplectic structure of the affine coadjoint orbit

where $\widetilde{\mathfrak{g}}_\mu$ is the isotropy algebra of μ respect to the affine coadjoint action associated with σ.

An example: Cotangent-lifted action with a magnetic term

- $L = T^*Q$
- $\pi : T^*Q \to Q$ the canonical projection
- G a Lie group acting properly on Q and its contagent lift on T^*Q
- $\omega = \Omega_Q - \pi^*\beta$ where $\beta \in \Omega^2(Q)$ is closed and G-invariant, and there exists $\phi : Q \to \mathfrak{g}^*$ such that

$$i_{\xi_Q}\beta = d\phi^\xi,\ \forall \xi \in \mathfrak{g}$$

In this context, $(T^*Q, \omega = \Omega_Q - \pi^*_Q\beta, G, J)$ is a Hamiltonian G-space, where the (non-equivariant) momentum map J is given by

$$J = J_Q - \phi \circ \pi_Q,$$

and its associated one-cocycle $\sigma : G \to \mathfrak{g}^*$ is defined

$$\begin{aligned}\sigma(g) &:= J(g \cdot \alpha_q) - \mathrm{Ad}^*_{g^{-1}}(J(\alpha_q))\\ &= \mathrm{Ad}^*_{g^{-1}}(\phi(q)) - \phi(g \cdot q).\end{aligned}$$

Fix $\alpha_q \in T^*Q$. The symplectic normal space of (T^*Q, ω, G, J) at α_q is described by

$$(V, \omega_V) \simeq (\widetilde{V}_\mu \oplus T^*B, \widetilde{\omega}_\mu(\mu) + \Omega_B)$$

where $B = ((\mathfrak{g}_q \cap \widetilde{\mathfrak{g}}_\mu) \cdot \alpha_q)^\circ \cap S$ and S is a complement of $\mathfrak{g} \cdot q$ at T_qQ.

Forthcoming Research

The next step that we propose to develop is the description of the Marle-Guillemin-Sternberg normal form of the Lagrangian fibration, using that description of the symplectic normal space.

References

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