

Finite Group Action in Gauge Theory

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I. Finite Group Action and Donaldson Invariant

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I. Finite Group Action and Donaldson Invariant

I.1 Review of Donaldson Invariant

- M : a simply connected smooth 4-manifold with positive definite intersection form.
- $E \longrightarrow M$: $SU(2)$ -bundle with $c_2(E) = -1$.
- A connection $\nabla \in \mathfrak{C}(E)$: self-dual if $*R^\nabla = R^\nabla$.

$\implies R^\nabla$: harmonic

∇ : minimize the Yang-Mills action

$$\mathcal{Y}_M(\nabla) = \frac{1}{2} \int_M \|R^\nabla\|^2 dvol.$$

- The gauge group \mathfrak{g} of bundle automorphisms of E acts on the space \mathcal{SD} of self-dual connections.

$\mathfrak{M}(E) = \mathcal{SD}/\mathfrak{g}$: the moduli space of self-dual connections.

By perturbation or generic metric and compactification.

Theorem I.1.1. $\overline{\mathfrak{M}}(E)$: a smooth, orientable, 5-manifold with λ singular points each of which has a cone nbd on $\mathbb{C}\mathbb{P}^2$, where $\lambda = \text{rank } H^2(M; \mathbb{Z})$.

Theorem I.1.2. If $H^2(M; \mathbb{Z}) = \langle a_1, \dots, a_\lambda \rangle$, then

$$M \simeq \prod_1^\lambda \mathbb{C}\mathbb{P}^2.$$

Theorem I.1.3.

(1) *The positive definite intersection form*

$$\omega(M) \simeq (1) \oplus \cdots \oplus (1).$$

(2) \exists *Non smoothable topological 4-manifold.*

Ex. nE_8

(3) \exists *Exotic smooth structures on \mathbb{R}^4 .*

(4) *Nondecomposability, $M_1 \# M_2$ ($b_2^+(M_i) > 0$, $i = 1, 2$)*

I.2. Obstruction for G -moduli space

- Let G be a finite group.

Suppose that G acts smoothly, semi-freely, isometrically, orientation-preservingly on a 1-conn., closed, smooth 4-manifold M with a p.d.i.f.

Let $\pi : E \longrightarrow M$: a G -bundle with $c_2(E) = -1$,
 $M^G = F := \{P_i\}_{i=1}^{n_1} \amalg \{T^{\lambda_i}\}_{i=1}^{n_2}$, G acts trivially on $E|_F$.

- $\overline{\mathfrak{M}}(E)$ may not be a G -manifold, but G -space. Transform the G -space $\overline{\mathfrak{M}}(E)$ into a smooth G -manifold with some singularities.

Theorem I.2.1. *\exists a Baire set in the G -invariant metrics such that $\widehat{\mathfrak{M}}^G$ is a smooth manifold in the $\widehat{\mathfrak{M}}$ of irreducible self-dual connections.*

For each $\nabla \in \mathfrak{M}^G$, G -invariant fundamental elliptic complex :

$$0 \longrightarrow \Omega^0(\mathfrak{g}_E) \xrightarrow{d^\nabla} \Omega^1(\mathfrak{g}_E) \xrightarrow{d^\nabla} \Omega^2(\mathfrak{g}_E) \longrightarrow 0 \quad .$$

Associate a Dirac operator

$$\begin{array}{ccc} \Gamma(V_+ \otimes V_- \otimes \mathfrak{g}_\mathbb{C}) & \xrightarrow{D} & \Gamma(V_- \otimes V_- \otimes \mathfrak{g}_\mathbb{C}) \\ & \searrow \nabla & \nearrow C \\ & \Gamma(\Lambda^1 \otimes V_+ \otimes V_- \otimes \mathfrak{g}_\mathbb{C}) & \end{array}$$

For each $g \in G$,

$$\text{ind}_g(D) = (-1)^m \frac{\text{ch}_g(j^* \sigma(D)) \text{td}(T^g \otimes \mathbb{C})}{\text{ch}_g(\Lambda_{-1} N^g \otimes \mathbb{C})} [TX^g],$$

$$m = \dim M^g,$$

$$j : M^g \longrightarrow M,$$

$$N^g = N(M, M^g) : \text{normal bundle.}$$

- Perturb the Fredholm G -map $\Psi : \mathcal{C}/\mathfrak{g} \longrightarrow \mathcal{C} \times \Omega_-^2(\mathfrak{g}_E)/\mathfrak{g}$ given by $\Psi([\nabla]) = [\nabla, R_-^\nabla]$ to transverse the zero section.

The Kuranishi map Ψ is locally equivalent to the sum of a G -equivariant linear map and a nonlinear G -equivariant map in finite dimensional range.

- Now we assume that $G = \mathbb{Z}_2 = \langle h \rangle$ is the group of order 2.

Proposition 1.2.2. *Let $A = \chi(F)$, $\nabla \in \mathfrak{M}(E)^G$ and $h(\nabla) = g(\nabla)$ for some $g \in \mathfrak{g}$.*

(i) *If $(hg)^2 = I$, $\nabla \in \widehat{\mathfrak{M}}$, then*

$$\begin{cases} \dim H_{\nabla_+}^1 - \dim H_{\nabla_+}^2 = \frac{1}{4}(10 + 3A) \\ \dim H_{\nabla_-}^1 - \dim H_{\nabla_-}^2 = \frac{1}{4}(10 - 3A) \end{cases},$$

where $H_{\nabla_{\pm}}^$ is (± 1) -eigen sp. of (hg) ,*

(ii) *If $(hg)^2 = -I$, $\nabla \in \widehat{\mathfrak{M}}(E)^G$, then*

$$\begin{cases} \dim H_{\nabla_+}^1 - \dim H_{\nabla_+}^2 = \frac{1}{4}(10 + A) \\ \dim H_{\nabla_-}^1 - \dim H_{\nabla_-}^2 = \frac{1}{4}(10 - A) \end{cases},$$

where $H_{\nabla_{\pm}}^$ is (± 1) -eigen space of $(hg)^2$,*

(iii) If ∇ is reducible, then

$$\begin{cases} \dim H_{\nabla_+}^1 - \dim H_{\nabla_+}^2 = \frac{1}{4}(14 + A) \\ \dim H_{\nabla_-}^1 - \dim H_{\nabla_-}^2 = \frac{1}{4}(10 - A) \end{cases} ,$$

where $H_{\nabla_{\pm}}^*$ is (± 1) -eigen space of $g_1 h g g_2$ for some $g_i \in \Gamma_{\nabla}$.

- Apply a G -transversality tech. of T. Petrie.

$$X = \mathfrak{M}(E)^G,$$

$$X_0 = (\text{End}(\mathfrak{M}(E)) \cup \text{ncd of reducible con. in } \mathfrak{M}(E)) \cap X,$$

For $\nabla \in X$, \exists a fiber bundle $V \longrightarrow X$ with fiber $V_{\nabla} = \text{Hom}_G^S(H_{\nabla_-}^1, H_{\nabla_-}^2)$

: the space of surjective G -homomorphisms

: a Stiefel manifold with

$$\pi_i(V_{\nabla}) = \begin{cases} \mathbb{Z} & \text{if } (hg)^2 = -I \text{ and } i = 2, \\ 0 & \text{if } (hg)^2 = I \text{ and } i \leq 3. \end{cases}$$

Theorem I.2.3.

(i) *To perturb Ψ to be G -transversal throughout of $\mathfrak{M}(E)^G$ there are obstruction classes $\Theta_3(\Psi) \in H^3(X, X_0; \mathbb{Z} = \pi_2(V_{\nabla}))$.*

(ii) *If the obstruction cohomology classes $\Theta_3(\Psi) = 0$, then we have a smooth G -manifold $\mathfrak{M}(E)$ of dimension 5 with λ singular points each of which has a cone nbd of $\mathbb{C}\mathbb{P}^2$, where $\lambda = \text{rank}H^2(M; \mathbb{Z})$.*

I.3. Involution and Donaldson invariant

- X : a closed, 1-conn. smooth 4-manifold with an orientation-preserving smooth involution σ .

$X^\sigma = F$: a 2-dim submanif.

$E \longrightarrow X$: $SU(2)$ vector bundle, $-c_2(E)$: even.

$p : X \rightarrow X/\sigma = X'$: projection, $F' = p(F)$.

- For a generic σ -invariant metric on X , let $\mathfrak{M}(E)^\sigma$: the moduli sp. of σ -invariant self-dual connections

Theorem I.3.1. \exists 1-1 correspondence between $\mathfrak{M}(E)^\sigma$ and $\mathfrak{M}(E')$, where $\mathfrak{M}(E')$: the moduli sp. of self-dual conn. on $E' \rightarrow X'$.

Let $\dim \mathfrak{M}^\sigma(E) = \dim \mathfrak{M}(E') = 2d$.

For the coupled SU(2)-bundle $E^* \rightarrow \overline{\mathfrak{M}}^\sigma(E) \times X$,

$$\begin{array}{ccc} \text{the Donaldson } \mu : H_2(X) & \xrightarrow{c_2(E^*)/} & H^2(\overline{\mathfrak{M}}^\sigma(E)) \\ & \searrow & \downarrow PD \\ & & H_{2d-2}(\overline{\mathfrak{M}}^\sigma(E)). \end{array}$$

Donaldson invariant is defined by

$$D^\sigma : H_2(X)^\sigma \times \cdots \times h_2(X)^\sigma \longrightarrow \mathbb{Z}$$

$$D^\sigma(\alpha_1, \cdots, \alpha_d) = \#[\overline{\mathfrak{M}}^\sigma(E) \cap \mu(\alpha_1) \cap \cdots \cap \mu(\alpha_d)].$$

Theorem I.3.2. (Wang) *Let $\alpha_1, \cdots, \alpha_d \in H_2(X; \mathbb{Z})^\sigma$ and $p_*(\alpha_i) = 2\beta_i \in H_2(X'; \mathbb{Z})$, $i = 1, \cdots, d$. Then $D^\sigma(\alpha_1, \cdots, \alpha_d) = D'(\beta_1, \cdots, \beta_d)$, where $D' : H_2(X') \times \cdots \times H_2(X') \longrightarrow \mathbb{Z}$: the Donaldson invariant on the quotient $E' \longrightarrow X'$.*

II. Finite Group Action and Seiberg-Witten Invariant

II.1. Review of Seiberg-Witten Invariant

- X : a closed, oriented, smooth 4-manifold, $b_2^+(X) > 1$.
 $L \rightarrow X$: $U(1)$ -bundle with $c_1(L) = c_1(X) \pmod{2}$.
 W^\pm : twisted spinor bundles assoc. to L .
Clifford multi. $W^+ \otimes T^*X \rightarrow W^-$
 $\tau : W^+ \times W^+ \rightarrow \text{End}(W^+)_0$
given by $\tau(\phi, \phi) = (\phi \otimes \bar{\phi}^t)_0$
: traceless endomorphism of W^+ .
- Levi-Civita conn. on X and a conn. A on L induce a Dirac operator
 $D_A : \Gamma(W^+) \rightarrow \Gamma(W^-)$.
Seiberg-Witten (SW) equations :

$$(\star) \quad \begin{cases} D_A \phi = 0 \\ F_A^+ = -\tau(\phi, \phi) \end{cases} .$$

- Gauge group $c^\infty(X, U(1))$ of L acts on $\mathcal{SW}(L) = \{(A, \phi) | (\text{⌘})\}$.
 $\mathfrak{M}(L) = \mathcal{SW}(L) / c^\infty(X, U(1))$: the moduli space associated to the spin^c structure L on X .
- For a generic metric on X , $\mathfrak{M}(L)$: a compact orientable manifold with $\dim d = \frac{1}{4}[c_1(L)^2 - (2\chi + 3\sigma)]$.
- Fix a point $x_o \in X$, the evaluation map $\varrho : c^\infty(X, U(1)) \longrightarrow U(1)$ induces a $U(1)$ -bundle
 $E = (\mathcal{SW}(L) \times \mathbb{C}) / c^\infty(X, U(1)) \longrightarrow \mathfrak{M}(L)$.

$$\mathcal{SW}(L) = \langle c_1(E)^s, \mathfrak{M}(L) \rangle$$

: the Seiberg-Witten invariant of L , where $d = 2s$.

Theorem II.1.1.

- (1) \exists finite number of spin^c str. L for which $\text{SW}(L) \neq 0$.
- (2) $X = X_1 \# X_2$, $b_2^+(X_i) > 0$, $i = 1, 2$,
then $\text{SW}(L) = 0$ for spin^c str. L on X .
- (3) $\text{SW}(L)$: indep. on the metrics on X , dep. only on $c_1(L)$.
- (4) f : a self-diffeo. of X , $\text{SW}(L) = \pm \text{SW}(f^*L)$.
- (5) If X admits a metric of positive scalar curv.,
then $\text{SW} \equiv 0$ on X .
- (6) If X : symplectic, then $\text{SW}(K_X) = \pm 1$.
- (7) Thom conjecture is true.

II.2. Finite Group Action and spin^c structure

Let $\pi_1(X)$: finite, G : a finite group.

Theorem II.2.1. *Let G act smoothly, freely on X , $\text{SW}_X \neq 0$. Then X/G cannot be decomposed as a smooth $X_1 \# X_2$ with $b_2^+(X_i) > 0, i = 1, 2$.*

Theorem II.2.2. *X : symplectic, $c_1(X)^2 > 0, b_2^+(X) > 3$. If σ : free anti-sym. involution on X , then $\text{SW} \equiv 0$ on X/σ and X/σ : not symplectic.*

Theorem II.2.3. *$\text{SW}_X \neq 0$ on $X, b_2^+(X) > 1$. If Y : negative definite intersection form and n_1, \dots, n_k : even integers such that $4b_1(Y) = 2n_1 + \dots + 2n_k + n_1^2 + \dots + n_k^2, \pi_1(Y)$ has a nontrivial finite quotient. Then $X \# Y$ has a nontrivial SW-invariant but does not admit any symplectic str.*

Theorem II.2.4. X : symplectic, σ : anti-sympl. involution

$$X^\sigma = \coprod_{\lambda} \Sigma_{\lambda}$$

: lagrangian surfaces, $\text{genus}(\Sigma_{\lambda}) > 1$ for some λ , $b_2^+(X/\sigma) > 1$.
Then X/σ has vanishing SW-invariants.

Example II.2.5. $X = (\Sigma_g \times \Sigma_g, w \oplus w)$. $f : \Sigma_g \longrightarrow \Sigma_g$: involution
 $f^*w = -w$, $\sigma_f : \Sigma_g \times \Sigma_g \longrightarrow \Sigma_g \times \Sigma_g$, $\sigma_f(x, y) = (f^{-1}(y), f(x))$.
Then σ_f : anti-symp. involution, $(\Sigma_g \times \Sigma_g)^{\sigma_f} = \text{graph}(f) \simeq \Sigma_g$.
By Hirzebruch signature thm.,

$$b_2^+(X/\sigma_f) = \frac{1}{2}(b_2^+(X) - 1) = g^2 > 1 \quad \text{if } g > 1.$$

Theorem II.2.6. X : Kähler surf., $b_2^+(X) > 3$,

$H_2(X; \mathbb{Z})$ has no 2-torsion.

$\sigma : X \rightarrow X$: anti-holom. involution,

$X^\sigma = \Sigma$ has genus > 0 , $[\Sigma] \in 2H_2(X; \mathbb{Z})$.

If $K_X^2 > 0$ or $K_X^2 = 0$ and $g(\Sigma) > 1$, then $\text{SW} \equiv 0$ on X/σ .

- \mathbb{Z}_p acts on X , p : prime, $H_1(X, \mathbb{R}) = 0$, $b_2^+(X) > 1$

Theorem II.2.7. (Fang) Supp. \mathbb{Z}_p acts trivially on $H^{2+}(X, \mathbb{R})$.

L : \mathbb{Z}_p -equivariant spin^c str. on X , the equivariant Dirac operator

$D_A : \Gamma(W^+) \rightarrow \Gamma(W^-)$ has the form

$$\text{ind } \mathbb{Z}_p(D_A) = \sum_{j=0}^{p-1} k_j t^j \in R(\mathbb{Z}_p) = \frac{\mathbb{Z}[t]}{(t^p = 1)}.$$

$$\Rightarrow \text{SW}(L) = 0 \pmod{p}$$

if $k_j \leq \frac{1}{2}(b_2^+(X) - 1)$, $j = 0, 1, \dots, p-1$.

Theorem II.2.8. (Baldrige) *If X is a smooth, closed, oriented 4-manifold with $b_2^+(X) > 1$, and admits a circle S^1 action with $X^{S^1} \neq \emptyset$, then $\text{SW} \equiv 0$ on X .*

III. Finite Group Action and Gromov-Witten Invariant.

III.1. Finite Group Action on Symplectic 4-Manifold

- (X, ω) : a closed symplectic 4-manifold. A finite group G acts pseudo-holomorphically, semifreely on X with a codim.2 fixed pt. set F .
- $p : X \rightarrow X/G = X'$: the projection.

$$\begin{array}{ccc} H^2(X; \mathbb{Z})^G & \xleftarrow{p^*} & H^2(X'; \mathbb{Z}) \\ PD \downarrow & & \downarrow |G| \cdot PD \\ H_2(X; \mathbb{Z})^G & \xrightarrow{p_*} & H_2(X'; \mathbb{Z}) \end{array}$$

commutes, where $|G|$: the order of G .

- Let $A' \in H_2(X'; \mathbb{Z})$, $A' = PD(\alpha') \in H_2(X'; \mathbb{Z})$

$$L' \rightarrow X' : U(1) \text{ vector bundle } c_1(L') = \alpha'$$

$$L \equiv p^*L' \rightarrow X, A = PD(c_1(L)).$$

Then $A \in H_2(X; \mathbb{Z})^G$ and $p_*(A) = |G|A'$.

- Let J be G -invariant, ω -compatible almost complex st. on X . ω', J' : push downs of ω, J resp. on X' .

$$\begin{aligned} p_* : H &= \text{im}(PD \circ p^*) \subset H_2(X; \mathbb{Z})^G \\ &\rightarrow H' = \text{im}(|G|PD) \subset H_2(X'; \mathbb{Z}) \end{aligned}$$

is an isomorphism.

For $A \in H$, $p_*(A) = |G|A'$, $F = X^G$, $F' = p(F) \subset X'$,

$$\begin{aligned} & \overline{\mathfrak{M}}_{0,k}(X, F : A, J)^G \\ &= [\{u : (C, x_1, \dots, x_k) \rightarrow (X, F; J) \mid u \\ & \quad : J - \text{holo. stable map represents } A \text{ relative to } F, \\ & \quad \text{arithmetic genus of } C = 0, u(C) : G - \text{inv.}, \\ & \quad k \text{ marked points}\} / \text{Aut}(u)]. \end{aligned}$$

$$\begin{aligned} & \overline{\mathfrak{M}}_{0,k}(X', F'; A', J') \\ &= [\{u' : (C, x_1, \dots, x_k) \rightarrow (X', F'; J') \mid u' : J' - \text{holo.} \\ & \quad \text{stable map represents } A' \text{ relative to } F', \\ & \quad \text{arithmetic genus of } C = 0, k \text{ marked points}\} / \text{Aut}(u')]. \end{aligned}$$

Theorem III.1.1.

(i) *There is a homeomorphism*

$$\Psi : \overline{\mathfrak{M}}_{0,k}(X, F; A, J)^G \rightarrow \overline{\mathfrak{M}}_{0,k}(X', F'; A', J')$$

which is an orientation preserving diffeomorphism on each corresponding strata.

(ii)

$$\begin{aligned} & \dim (\overline{\mathfrak{M}}_{0,k}(X, F; A, J)^G) \\ &= \dim (\overline{\mathfrak{M}}_{0,k}(X', F'; A', J')) \\ &= d'. \end{aligned}$$

$$\begin{array}{ccccc} & & (C, x_1, \dots, x_k) & & \\ & \swarrow \theta & \downarrow \phi' & \searrow u & \\ (C, x_1, \dots, x_k) & \xrightarrow{u} & & \xrightarrow{u} & (X, F) \\ & \downarrow \phi & & & \downarrow p \\ & & (C, x_1, \dots, x_k) & & \\ & \swarrow \theta' & \downarrow & \searrow u' & \\ (C, x_1, \dots, x_k) & \xrightarrow{u'} & & \xrightarrow{u'} & (X', F') \end{array}$$

III.2. Relative Gromov-Witten Invariant

The evaluation map

$$\text{ev}_k : \overline{\mathfrak{M}}_{0,k}(X, F; A, J)^G \rightarrow X^k$$

$$\text{ev}_k([C, x_1, \dots, x_k; u]) = (u(x_1), \dots, u(x_k)).$$

$\text{im}(\text{ev}_k) \subset X^k$: a d' -dim. pseudo-cycle whose boundary has at most dim. $d' - 2$.

The relative G -invariant Gromov-Witten invariant

$$\Phi_{A,k}^{F,G} : H_d(X^k) \rightarrow \mathbb{Q}$$

is given by the integral

$$\Phi_{A,k}^{F,G}(D) = \int_{\overline{\mathfrak{M}}_{0,k}(X, F; A, J)^G} \text{ev}_k^* PD(D) = (\text{im } \text{ev}_k) \cdot D$$

: intersection number in X^k , where $d + d' = 4k$.

Similarly, the evaluation map

$$\text{ev}_k' : \overline{\mathfrak{M}}_{0,k}(X', F'; A', J') \rightarrow X'^k$$

is given by

$$\text{ev}_k'([C, x_1, \dots, x_k; u']) = (u'(x_1), \dots, u'(x_k)).$$

The relative Gromov-Witten invariant on the quotient

$$\Phi_{A',k}^{F'} : H_d(X'^k) \rightarrow \mathbb{Q}$$

is defined by the integral

$$\Phi_{A',k}^{F'}(D') = \int_{\overline{\mathfrak{M}}_{0,k}(X', F'; A', J')} \text{ev}_k'^* PD(D') = (\text{im}(\text{ev}_k')) \cdot D$$

: intersection number in X'^k , where $d' + d = 4k$.

- If $2c_1(X')A' = 2 + 2k + 2F' \cdot A'$, then the invariant is the degree of the ev_k' which is the number of J' -holomorphic curves representing the homology A' meeting generic k distinct points in F' tangentially.

Theorem III.1.2. *Let $D \in H \subset H_d(X^k; \mathbb{Z})^G$ and $D' \in H' \subset H_d(X'^k; \mathbb{Z})$ such that $p_*(D) = |G|^k D'$, then*

$$\Phi_{A,k}^{F,G}(D) = \Phi_{A',K}^{F'}(D').$$

$$\text{i.e., } D \in H_d(X^k; \mathbb{Z})^G = \sum_{d_1 + \dots + d_k = d} H_{d_1}(X)^G \times \dots \times H_{d_k}(X)^G$$

$\exists D_{d_i} \in H_{d_i}(X)^G, i = 1, \dots, k$ such that

$$D = \sum_{d_1 + \dots + d_k = d} D_{d_1} \times \dots \times D_{d_k}.$$

$$p_*(D_{d_i}) = |G| D_{d_i}', i = 1, \dots, k,$$

$$\begin{aligned} p_*(D) &= p_*\left(\sum_{d_1 + \dots + d_k = d} D_{d_1} \times \dots \times D_{d_k}\right) \\ &= \sum_{d_1 + \dots + d_k = d} p_*(D_{d_1}) \times \dots \times p_*(D_{d_k}) \\ &= |G|^k p_*(D'). \end{aligned}$$

$\Phi_{A,k}^{F,G}(D) = (\text{im}(\text{ev}_k)) \cdot D$: intersection number in X^k .

III.3. Example

- Recall the number $R(d)$ of rational curves of the degree d in $\mathbb{C}\mathbb{P}^2$.

$$\dim \overline{\mathfrak{M}}_{0,k}(\mathbb{C}\mathbb{P}^2, d[S^2]) = 2(3d - 1 + k)$$

If $k = 3d - 1$, then $\dim \overline{\mathfrak{M}}_{0,k}(\mathbb{C}\mathbb{P}^2, d[S^2]) = 4(3d - 1)$,

$$\text{ev}_k : \overline{\mathfrak{M}}_{0,k}(\mathbb{C}\mathbb{P}^2, d[S^2]) \rightarrow (\mathbb{C}\mathbb{P}^2)^k.$$

$$\begin{aligned} R(D) &= \int_{\overline{\mathfrak{M}}_{0,k}(\mathbb{C}\mathbb{P}^2, d[S^2])} \prod_{i=1}^k \text{ev}_k^*(c_1(\mathcal{O}(1)_i^2)) = \text{deg}(\text{ev}_k) \\ &= \begin{cases} \sum_{d_1+d_2=d} R(d_1)R(d_2)d_1^2d_2 \left[k_2 \binom{3d-4}{3k_1-2} - k_1 \binom{3d-4}{3k_1-1} \right], & d \geq 2 \\ 1, & d = 1 \end{cases} \end{aligned}$$

computed by the WDVV-equations for the potential.

- For $\mathbb{CP}^1 \subset \mathbb{CP}^2$,

$$\dim \overline{\mathfrak{M}}_{0,k}(\mathbb{CP}^2, \mathbb{CP}^1; d\mathbb{CP}^1) = 2(2d - 1 + k)$$

$$\text{If } k = 2d - 1, \text{ then } = 4(2d - 1)$$

The number of rational curves in \mathbb{CP}^2 of degree d relative to \mathbb{CP}^1 passing through generic $(2d - 1)$ points in \mathbb{CP}^1 is

$$\begin{aligned} R'(d) &= \int_{\overline{\mathfrak{M}}_{0,k}(\mathbb{CP}^2, \mathbb{CP}^1; d\mathbb{CP}^1)} \prod_{i=1}^k \text{ev}_k'^*(c_1(Q(1)_i)^2) \\ &= \text{degree of } [\text{ev}_k' : \overline{\mathfrak{M}}_{0,k}(\mathbb{CP}^2, \mathbb{CP}^1; d\mathbb{CP}^1) \rightarrow (\mathbb{CP}^2)^k] \end{aligned}$$

- $\sigma : \mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$: the involution given by $\sigma(z_1, z_2) = (z_2, z_1)$.

Then $(\mathbb{CP}^1 \times \mathbb{CP}^1)^\sigma = \Delta \simeq \mathbb{CP}^1$, and $(\mathbb{CP}^1 \times \mathbb{CP}^1)/\sigma \simeq \mathbb{CP}^2$.

Commutative diagram

$$\begin{array}{ccc}
 H^2(\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1)^\sigma & \xleftarrow{p^*} & H^2(\mathbb{C}\mathbb{P}^2) \\
 PD \downarrow & & \downarrow 2 \cdot PD \\
 H_2(\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1)^\sigma & \xrightarrow{p_*} & H_2(\mathbb{C}\mathbb{P}^2) .
 \end{array}$$

Let $A, B \in H_2(\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1)$, $C \in H_2(\mathbb{C}\mathbb{P}^2)$ be generators.

$$\begin{aligned}
 p_*(\Delta) &= p_*(A, B) = 2C, \\
 \dim \overline{\mathfrak{M}}_{0,k}(\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1, \Delta; d(\Delta))^\sigma & \\
 &= 2C_1(\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1) \cdot d(\Delta) + 2k - 2 - 2 \cdot \Delta \cdot d(\Delta) \\
 &= 2(2d - 1 + k).
 \end{aligned}$$

For $D \in H.[(\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1)^k]^\sigma$, $p_*(D) = 2^k D' \in H.(\mathbb{C}\mathbb{P}^2)^k$

The relative Gromov-Witten invariant

$$\begin{aligned}
 \Psi_{d(\Delta),k}^{\Delta,\sigma}(D) &= \Psi_{dC,k}^C(D') \\
 &= \Psi_{dC,k}^C(\text{pt}, \dots, \text{pt}) \text{ if } k = 2d - 1 \\
 &= R'(d) \\
 &= \deg[\text{ev}'_k : \overline{\mathfrak{M}}_{0,k}(\mathbb{C}\mathbb{P}^2, \mathbb{C}\mathbb{P}^1, dC) \rightarrow (\mathbb{C}\mathbb{P}^2)^k].
 \end{aligned}$$