The Splitting Problem in Riemannian and Lorentzian Geometry

Lectures on Lorentzian Geometry
Universidad de Granada

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Plan of the Lecture

- Introduction
- Riemannian Splitting Theorem
  - Cheeger-Gromoll’s Proof
  - Eschenburg-Heintze’s Proof
- Preliminaries on Lorentzian Geometry
- Lorentzian Splitting Theorem
  - Preliminary results: super-harmonicity; nice neighborhoods, convexity
  - Proof of the theorem: 6 steps
- Open Problem: Bartnik’s Conjecture
Introduction
Introduction

Rigidity in Geometry:

- Sometimes, it is useful to compare the geometry of a general manifold $M$ with that of a simply connected model space $M_K$ of constant curvature $K$.

- Typically, under certain “strict” curvature bounds in terms of $K$, $M$ will retain particular geometrical properties of $M_K$.

- Once this has been established, it is usually possible to conclude that $M$ retains topological properties of $M_K$ as well.

- But, what happen when one relaxes the condition of “strict” curvature inequality to “weak” curvature inequality?
The conclusion may not hold any more. For example, observe the big difference between the topologies of:

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However, a conclusion which becomes false when one relaxes the condition of “strict” inequality to “weak” inequality usually can be shown to fail under very special circumstances!
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However, a conclusion which becomes false when one relaxes the condition of “strict” inequality to “weak” inequality usually can be shown to fail under very special circumstances!

**Prototype Rigidity Result:**

*If* \( M \) *satisfies a “weak” curvature inequality, and the geometric restriction derived from the “strict” curvature inequality does not hold, \( M \) must be “very special”.*
**Theorem:** (Gromoll, Meyer: *Ann. of Math*, 90, 75-90, 1969)

A complete Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) such that \(\text{Ric}(v, v) > 0\) for all \(v \in TM\) is connected at infinity.

Suppose that \(\text{Ric}(v, v) \geq 0\) and that \((M, g)\) fails to be connected at infinity.

Since \(M\) is assumed to be complete, there exists a line (i.e. a complete unitary geodesic \(\gamma : \mathbb{R} \to (M, g)\) realizing the distance between any two of its points) joining any two ends of \(M\).

Then, Cheeger and Gromoll proved that \((M, g)\) must be isometric to a product manifold.
More precisely:

**Theorem:** (Cheeger, Gromoll: *J. Diff. Geom.*, 6, 119-128, 1971)

Suppose that the Riemannian manifold \((M, g)\), of dimension \(n \geq 2\), satisfies the following conditions:

1. \((M, g)\) is geodesically complete,
2. \(\text{Ric}(v, v) \geq 0\) for all \(v \in TM\),
3. \(M\) has a line.

Then \(M\) is isometric to the product \((M, g) \cong (\mathbb{R}^k \times M_1, g_0 \oplus g_1)\), \(k > 0\), with \(M_1\) containing no lines and \(g_0\) the standard metric on \(\mathbb{R}^k\).
In 1982, Yau proposed to obtain the Lorentzian analog of this result. As consequence:

**Theorem (Lorentzian Splitting):**

Suppose that the spacetime \((M, g)\), of dimension \(n > 2\), satisfies the following conditions:

1. \((M, g)\) is timelike geodesically complete or globally hyperbolic,
2. \(\text{Ric}(v, v) \geq 0\) for all timelike \(v \in TM\),
3. \(M\) has a timelike line.

Then \(M\) splits isometrically as \((M, g) \cong (\mathbb{R} \times M_1, -dt^2 \oplus g_1)\), where \((M_1, g_1)\) is a complete Riemannian manifold.
Riemannian Splitting Theorem
Theorem (Riemannian Splitting):

Suppose that the Riemannian manifold \((M, g)\), of dimension \(n \geq 2\), satisfies the following conditions:

1. \((M, g)\) is geodesically complete
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History of the Problem:

- In 1964 Topogonov obtained the splitting under the more restrictive assumption of nonnegative sectional curvature.
- The proof of the Topogonov’s result lies on the Triangle Comparison Theorem, which does not work for nonnegative Ricci manifolds.
- Cheeger and Gromoll wanted to extend the existing results on the fundamental group to these less restrictive manifolds...for which they needed a splitting theorem (Cohn-Vossen’36, dim= 2).
- The first proof of the theorem was obtained by these authors in 1971, and was simplified later by Eschenburg and Heintze (1984).
Some Previous Definitions:
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- Given a ray $\gamma$, the Busemann function (associated to $\gamma$) is defined as the limit

$$b_\gamma(\cdot) := \lim_{r \to \infty} (r - d(\cdot, \gamma(r))).$$

Busemann function $b_\gamma$ always exists ($< \infty$) and is continuous.
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Busemann function $b_\gamma$ always exists ($< \infty$) and is continuous.

- Given a line $\gamma$, there are two natural rays associated to $\gamma$:

$\gamma_+ := \gamma|_{[0, \infty)}$ and $\gamma_-(t) := \gamma(-t)$, $t \in [0, \infty)$. We will denote by $b_\pm$ the corresponding Busemann functions associated to $\gamma_\pm$. 
Given a ray $\gamma$, we say that $\alpha : [0, \infty) \to M$ is an *asymptote* to $\gamma$ if it is a ray which arises as limit of minimal geodesic segments from some $p$ to $\gamma(r_n)$. 
Riemannian Splitting Theorem

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- Asymptotes from some point are not necessarily unique.
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- Assume $\alpha : [0, \infty) \to M$ is an asymptote to $\gamma$. Using TI, one derives:

$$b(\alpha(t)) = t + b(\alpha(0)) \quad \forall t \in [0, \infty).$$
Riemannian Splitting Theorem

Cheeger and Gromoll’s Proof:
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Let $\gamma$ be the line from the hypotheses of the theorem. From TI,

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- Next, we establish the main ingredient of the proof: the sub-harmonic character of Busemann functions \( b_\pm \).

**Theorem:** *If the Ricci tensor is nonnegative then functions \( b_\pm \) are sub-harmonic.*
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Next, we establish the main ingredient of the proof: the sub-harmonic character of Busemann functions $b_\pm$.

**Theorem:** If the Ricci tensor is nonnegative then functions $b_\pm$ are sub-harmonic.

A function $f$ is sub-harmonic if, given any connected compact region $D$ in $M$ with smooth boundary $\partial D$, one has $f \leq h$ on $D$, for $h$ the continuous function on $D$, harmonic on int $D$, with $h |_{\partial D} \equiv f |_{\partial D}$. 
Proof of the theorem:

- Denote by $d_p(\cdot) := d(\cdot, p)$ the distance function to $p$. Then

$$\Delta d_p(q) \leq (n - 1)/d_p(q) \quad \text{for any } q \text{ outside the cut locus of } p.$$ 

- One is tempted to deduce that $b_\gamma(\cdot) = \lim_r (r - d_\gamma(r)(\cdot))$ should have nonnegative Laplacian $\Delta b_\gamma \geq 0$, thus sub-harmonic.

- But notice that $d_\gamma(r)$ is not differentiable on the cut locus of $\gamma(r)$, and so, $b_\gamma$ may not be differentiable anywhere.

- Even though $b_\gamma$ were differentiable almost everywhere with $\Delta b_\gamma \geq 0$, the conclusion is not clear at all.

- A sophisticated analysis of the behavior of the gradient near the points of non-differentiability is needed.

- The conclusion follows by taking sequences of approximations to $b_\gamma$ in pertinent regions. ☐
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Riemannian Splitting Theorem

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- Let $q \in \gamma$, $D$ connected compact region with $q \in \text{int}D$.
- Let $h_{\pm}$ continuous on $D$, harmonic on $\text{int}D$, $h_{\pm} \mid_{\partial D} = b_{\pm} \mid_{\partial D}$ (hence $h_+ + h_- = b_+ + b_- \leq 0$ on $\partial D$).
- From the Maximum Principle $h_+(q) + h_-(q) \leq 0 = b_+(q) + b_-(q)$.
- But $b_{\pm}$ sub-harmonic $\Rightarrow b_{\pm}(q) \leq h_{\pm}(q) \Rightarrow b_{\pm}(q) = h_{\pm}(q)$.
- Then, $b_{\pm} - h_{\pm}$ sub-harmonic, $b_{\pm} - h_{\pm} = 0$ on $\partial D$, $(b_{\pm} - h_{\pm})(q) = 0$ $\Rightarrow b_{\pm} = h_{\pm}$ on $D$. 

The Splitting Problem in Riemannian and Lorentzian Geometry – p.27/145
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- $|\text{grad } b_+| \equiv 1$, and the integral curves of $\text{grad } b_+$ are geodesics.

  - $|b_+(p) - b_+(q)| \leq d(p, q) \Rightarrow |\text{grad } b_+| \leq 1$

  - $|b_+(p) - b_+(q)| = d(p, q)$, $p, q \in \sigma$ from $p$ to $\gamma$.

  - $|\text{grad } b_+| = 1$, and $\sigma$ is integral curve of $\text{grad } b_+$ through $p$. 

The Splitting Problem in Riemannian and Lorentzian Geometry – p.29/145
Denote $N := \text{grad} b_+$. Then $\nabla_N N = 0$, and using that $b_+$ is harmonic, a direct computation gives

$$Ric(N) = \sum_{i=1}^{n-1} \langle R(E_i, N)N, E_i \rangle$$

$$= \sum_{i=1}^{n-1} \langle \nabla_{E_i} \nabla_N N - \nabla_N \nabla_{E_i} N - \nabla_{[E_i, N]} N, E_i \rangle$$

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- By the de Rham decomposition theorem, this map is isometry:

$$(b_+)^{-1}(0) \times \mathbb{R} \to M, \quad (p, t) \mapsto \exp(t \cdot \text{grad } b_+(p)).$$
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The conclusion follows by a finite induction on the lines of $M$. □
Remarks:

- This approach cannot be directly translated to the Lorentzian case because the D’alambertian operator is hyperbolic, not elliptic.
- There is an alternative approach which minimizes the use of the theory for elliptic operators.
- It is based on a direct application of the maximum principle together with a closer analysis of the geometry of the Busemann function associated to a line.
Riemannian Splitting Theorem

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- For any \( p, r \), we define \( b_{p,r}^\pm (x) := b_\pm(p) - r + d(x, \exp(rv)) \), where \( v \) is the direction of some asymptote from \( p \) to \( \gamma^\pm \).
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- \( b_{p,r}^\pm \) lower support functions of \( b_\pm \) at \( p \), which are \( C^\infty \) around \( p \); i.e.

  \[ b_{p,r}^\pm \leq b_\pm \text{ on } M, \quad b_{p,r}^\pm(p) = b_\pm(p). \]
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- From the nonnegative Ricci curvature hypothesis, one has

  \[ \Delta(b_{p,r}^+ + b_{p,r}^-) \geq -2(n - 1)/r. \]
Hopf-Calabi Max. Pple ’57:

Let $M$ be a connected Riemannian manifold and $f \in C^0(M)$. If for each $p \in M$ and any $\epsilon > 0$ there is a support function $f_{p,\epsilon}$ of $f$ at $p$ which is $C^2$ in a neighborhood of $p$ and satisfies $\Delta f_{p,\epsilon}(p) \geq -\epsilon$ then $f$ attains no maximum unless it is constant.
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- Hopf-Calabi Max. Pple ’57 ⇒ $b_+ + b_-$ attains no maximum unless it is constant.

- But $b_+ + b_-$ attains a maximum at $\gamma$; hence $b_+ + b_- \equiv 0$, and so, we have the sandwich

\[ b^+_p \leq b_+ = -b_- \leq -b^-_p, \quad \text{with “=” at } p. \]
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$$b_+^\pm \leq b_+ = -b_\_ \leq -b_\_^\pm, \text{ with } “=" \text{ at } p.$$  

$b_\pm$ is once differentiable at $p$, and $\text{grad } b_\pm(p) = \text{grad } b_\pm^\pm(p)$, which implies

$$|\text{grad } b_+| = 1.$$
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$b_\pm$ is once differentiable at $p$, and $\text{grad } b_\pm(p) = \text{grad } b^\pm_{p,r}(p)$, which implies

$$|\text{grad } b_+| = 1.$$

In particular, the asymptotes to $\gamma_\pm$ at any $p$ are uniquely determined and fit together to a line.
On the other hand, it can be proved that

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Thus, for any geodesic $c$, the functions $b^\pm \circ c$ admit support functions at any $t \in \mathbb{R}$ with arbitrarily small 2nd derivative at $t$.

Hence, the functions $b^\pm \circ c$ are convex by the (trivial 1-dim.) maximum principle.
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Therefore, \( b_+ = -b_- \) is convex and concave, thus constant along geodesics.
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Hence, the functions $b_{\pm} \circ c$ are convex by the (trivial 1-dim.) maximum principle.

Therefore, $b_+ = -b_-$ is convex and concave, thus constant along geodesics.

Hence, $b_+$ has totally geodesic level sets, and so, $N = \text{grad} \, b_+$ is a parallel vector field.
By the de Rham decomposition theorem, this map is isometry:

\[(b_{+})^{-1}(0) \times \mathbb{R} \rightarrow M, \quad (p, t) \mapsto \exp(t \cdot N(p)).\]

The conclusion follows by a finite induction on the lines of \(M\). \(\square\)
Preliminaries on Lorentzian Geometry
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- A *spacetime* is a pair $(M, g)$ with
  
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  M & \text{smooth manifold} \\
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  \[ \text{timelike if } g(v, v) < 0 \]
  \[ \text{lightlike if } g(v, v) = 0 \]
  \[ \text{spacelike if } g(v, v) > 0, \quad v = 0. \]
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\text{timelike} \text{ if } & \quad g(v, v) < 0 \\
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timelike if $g(v, v) < 0$
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- A smooth curve $\gamma : I \rightarrow M$ is
timelike if so is $\dot{\gamma}(s) \forall s$
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- A smooth curve \(\gamma : I \rightarrow M\) is timelike if so is \(\dot{\gamma}(s) \forall s\) and causal if so is \(\dot{\gamma}(s) \forall s\).

- Spacetimes are assumed time-oriented, i.e. they are endowed with a continuous, globally defined, timelike vector field \(X\).
Causal vector $v \in T_p M$ is

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\text{future-directed if } &\ g(v, X(p)) < 0 \\
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\end{align*}
\]

Causal tangent vectors $v \in T_p M$ are distributed in two cones, the future and the past one.

A causal curve $\gamma(s)$ is

\[
\begin{align*}
\text{future-directed} & \text{ if so is } \dot{\gamma}(s) \forall s \\
\text{past-directed} & \text{ if so is } \dot{\gamma}(s) \forall s.
\end{align*}
\]

Future-directed causal curves represent all the physically admissible trajectories for material particles and light rays in the universe.
Preliminaries

Causal Structure:
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- $q, p$ chronologically related, $q \ll p$, if they can be joined by a future timelike curve.

- *Chronological past of* $p$, $I^-(p)$, *is the set of events* $q$ *which are chronologically related to* $p$, $q \ll p$ (*analogous for* $I^+(p)$).
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- The chronological (resp. causal) past/future of $p$ defined w.r.t. an open set $U \subset M$ will be denoted by $I^{\pm}(p, U)$ (resp. $J^{\pm}(p, U)$).
Preliminaries

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- A spacetime is *causal* if it does not admit closed causal curves.
- A spacetime is *strongly causal* if it does not admit neither closed nor “almost closed” causal curves, i.e. for every \( p \in V \), there exists some neighborhood \( p \in U \subset V \) such that any causal curve with extremes inside \( U \) is totally contained in \( V \).
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$\iff$ it admits a *Cauchy hypersurface*: a topological hypersurface that is met exactly once by every inextensible timelike curve.
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- The edge of an achronal set $A \subset M$ is the set of points $p \in \overline{A}$ satisfying that every neighborhood $U$ of $p$ contains a timelike curve from $I^-(p, U)$ to $I^+(p, U)$ which does not meet $A$. 
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Properties:

1. $\overline{A} \setminus A \subseteq \text{edge}(A)$.
2. $A$ achronal is a topological hypersurface iff $A \cap \text{edge}(A) = \emptyset$.
3. $A$ achronal is a closed topological hypersurface iff $\text{edge}(A) = \emptyset$. 
Preliminaries

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$\Rightarrow$ $S$ is a closed topological spacelike hypersurface.
Temporal Separation/Lorentzian Distance:

\[ d(p, q) = \begin{cases} 
0, & \text{if } C_{pq}^c = \emptyset \\
\sup \{ L(\alpha) = \int \sqrt{-g(\dot{\alpha}, \dot{\alpha})}, \alpha \in C_{pq}^c \}, & \text{if } C_{pq}^c \neq \emptyset.
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\end{cases} \]

Properties:

1. \( d(p, q) > 0 \iff p \in I^-(q) \iff q \in I^+(p) \).
2. \( d(p, p) = \infty \) if \( \exists \) piecewise smooth timelike curve joining \( p \) to itself; otherwise, \( d(p, p) = 0 \).
3. If \( 0 < d(p, q) < \infty \) then \( d(q, p) = 0 \); hence, \( d \) is not symmetric.
4. If \( p \leq q \leq r \) then \( d(p, q) + d(q, r) \leq d(p, r) \) (Reverse T.I.).
5. In general, $d$ is not continuous, but only lower semicontinuous, i.e. if $\{p_m\} \rightarrow p$ and $\{q_m\} \rightarrow q$ then

$$\lim_{m} \inf d(p_m, q_m) \geq d(p, q).$$
5. In general, \( d \) is not continuous, but only lower semicontinuous, i.e. if \( \{p_m\} \to p \) and \( \{q_m\} \to q \) then

\[
\liminf_m d(p_m, q_m) \geq d(p, q).
\]

Lorentzian Distance and Global Hyperbolicity:

6. If \((M, g)\) is globally hyperbolic and \( p \leq q \) then there exists some maximal geodesic joining \( p \) to \( q \) (Avez-Seifert).

7. If \((M, g)\) is globally hyperbolic, \( d \) is continuous and finite valued.
Busemann Function:
Preliminaries

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- The Busemann function $b : I[\gamma] \to \mathbb{R}$, $I[\gamma] = I^+(\gamma(0)) \cap I^-[\gamma]$, associated to a ray $\gamma$ defined below always exists:

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b_\gamma(\cdot) := \lim_{{r \to \infty}} (r - d(\cdot, \gamma(r)))
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The Splitting Problem in Riemannian and Lorentzian Geometry – p.92/145
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  b_\gamma (\cdot) := \lim_{r \to \infty} (r - d (\cdot, \gamma (r)))
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- Busemann function may be \(-\infty\) and discontinuous.

- From the RTI, one derives \( b (q) \geq b (p) + d (p, q), \forall p, q \in I[\gamma], \) \( p \leq q. \)

- The level sets \( \{ b = c \} \) are achronal in \( I[\gamma]. \)
Preliminaries

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An asymptote to $\gamma$ from $p \in I[\gamma]$ is a ray $\alpha : [0, \infty) \to M$ which arises as limit of maximal timelike geodesic segments from $p$ to $\gamma(r_n)$. 

The Splitting Problem in Riemannian and Lorentzian Geometry – p.95/145
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Asymptotes:

- An asymptote to $\gamma$ from $p \in I[\gamma]$ is a ray $\alpha : [0, \infty) \to M$ which arises as limit of maximal timelike geodesic segments from $p$ to $\gamma(r_n)$.
- Asymptotes need not be timelike nor unique.
- Assume $\alpha : [0, \infty) \to M$ is an asymptote to $\gamma$. Using RTI, one derives:

  $$b(\alpha(t)) = t + b(\alpha(0)) \quad \forall t \in [0, \infty).$$
Lorentzian Splitting Theorem
Lorentzian Splitting Theorem

In 1982, Yau posed the problem of obtaining the Lorentzian analogue of the Cheeger-Gromoll Splitting theorem:

**Yau’s Conjecture:** (Ann. Math Studies, 102, Princeton, 1982)

Suppose that the spacetime $(M, g)$, of dimension $n > 2$, satisfies the following conditions:

1. $(M, g)$ is timelike geodesically complete,
2. $\text{Ric}(v, v) \geq 0$ for all timelike $v \in TM$,
3. $M$ has a timelike line.

Then $M$ splits isometrically as $(M, g) \cong (\mathbb{R} \times M_1, -dt^2 \oplus g_1)$, where $(M_1, g_1)$ is a complete Riemannian manifold.
History of the Problem

- This conjecture has involved multiple leading authors: Beem, Ehrlich, Markovsen, Galloway, Eschenburg, Newman...
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This conjecture has involved multiple leading authors: Beem, Ehrlich, Markovsen, Galloway, Eschenburg, Newman...

In 1984 Galloway solved the problem for the spatially closed case, under some additional hypotheses. Concretely:

Let \((M, g)\) be a spacetime which contains a compact Cauchy surface and which satisfies the timelike convergence condition. Assume \((M, g)\) contains a past and future complete timelike curve. If for each \(p \in M\) all null geodesics through \(p\) enter \(I^+(p)\) and \(I^-(p)\), then \(M\) is isometric to a product \(\mathbb{R} \times S\).

He applied maximal surfaces techniques from results in Gerhardt’83, Bartnik’84.
Afterwards Beem, Ehrlich, Markovsen and Galloway solved the problem by assuming global hyperbolicity instead of timelike geodesic completeness, and inequality $K \leq 0$ instead of $Ric \geq 0$.

Global hyperbolicity is not a big restriction, and, in certain sense, is more natural condition than timelike geodesic completeness.

The sectional curvature inequality is significatively more restrictive than that of Ricci curvature, and does not admit a clear physical interpretation.

This curvature hypothesis is assumed in order to apply a Lorentzian adaptation of the Topogonov’s argument, via de Harris’ Lorentzian Triangle Comparison Theorem (1982).
History of the Problem

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- In 1988, Eschenburg solved the problem for $\text{Ric} \geq 0$, by assuming both, global hyperbolicity and timelike completeness.

It is a direct translation to the Lorentzian case of the Eschenburg and Heintze’s arguments.

The key point was the observation that the geometry of a neighborhood of the timelike line allows certain arguments to be successfully modified from $K \leq 0$ to $\text{Ric} \geq 0$. 
In 1989, Galloway removed the assumption of timelike completeness from Eschenburg’s work. He applied a result on the existence of maximal spacelike hypersurfaces due to Bartnik (1988).

This result provided a more natural and powerful approach to the key step in Eschenburg’s result, making redundant the timelike geodesic completeness hypothesis.
History of the Problem

- In 1990 Newman obtained a proof assuming timelike completeness instead of global hyperbolicity, and thus, solved the Yau’s Conjecture.

New techniques are not introduced, just adaptations of the previous ones in order to carefully study the behavior of an arbitrary spacetime in a tubular neighborhood of a timelike line.

He employed Galloway’s maximal surfaces techniques for the sake of completeness.
All these results may be summarized in the following simple statement of the Lorentzian Splitting Theorem (containing Yau’s Conjecture):

**Theorem (Lorentzian Splitting):**

Suppose that the spacetime \((M, g)\), of dimension \(n > 2\), satisfies the following conditions:

1. \((M, g)\) is either globally hyperbolic or timelike complete,
2. \(\text{Ric}(v, v) \geq 0\) for all timelike \(v \in TM\),
3. \(M\) has a timelike line.

Then \(M\) splits isometrically as \((M, g) \cong (\mathbb{R} \times M_1, -dt^2 \oplus g_1)\), where \((M_1, g_1)\) is a complete Riemannian manifold.
History of the Problem

For simplicity reasons, we will study the following version of the Lorentzian Splitting Theorem:

**Theorem (Lorentzian Splitting):**

Suppose that the spacetime \((M, g)\), of dimension \(n > 2\), satisfies the following conditions:

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Then \(M\) splits isometrically as \((M, g) \cong (\mathbb{R} \times M_1, -dt^2 \oplus g_1)\), where \((M_1, g_1)\) is a complete Riemannian manifold.
Previous Technical Results
The first lemma is a convexity result which states the super-harmonic character of Lorentian Busemann functions whenever they are smooth:

**Lemma:**

Assume $M$ obeys $\text{Ric}(v, v) \geq 0$ for all timelike $v \in TM$. Let $b$ be the Busemann function associated to a ray $\gamma$. If $b$ is smooth on an open set $U \subset I[\gamma]$ with unit timelike gradient then $\Delta b \leq 0$ on $U$. 
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**Lemma:**

Assume $M$ obeys $\text{Ric}(v, v) \geq 0$ for all timelike $v \in TM$. Let $b$ be the Busemann function associated to a ray $\gamma$. If $b$ is smooth on an open set $U \subset I[\gamma]$ with unit timelike gradient then $\Delta b \leq 0$ on $U$.

**Proof:**

- Assume by contradiction $\Delta b(p) = H > 0$ for some $p \in U$.
- Denote $\Sigma := \{b = c\} \cap U_0$, $c = b(p)$, $U_0$ certain neighborhood.
- Since $b_r \downarrow b$, it is $\Sigma \subset \{b_r \geq c\}$ for all $r \geq r_0$. 

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- Since $\nabla b$ is past-directed and unitary, the mean curvature of $\Sigma$ becomes $H_\Sigma = \Delta b$ along $\Sigma$.

- Choose some $q \in I^+(p) \cap U_0$ close enough to $p$ so that $H_\Sigma(x) \geq H/2$ for all $x \in \Sigma \cap I^-(q)$.

- Let $\Sigma'$ be a smooth spacelike hypersurface resulting from a small deformation of $\Sigma$ around $p$ such that:

  (i) $A = \Sigma' \setminus \Sigma \subset I^-(q)$.

  (ii) $A \cap I^-(p) \neq \emptyset$.

  (iii) $H_{\Sigma'}(x) \geq H/3$ for all $x \in A$.

- Since $b_r(p) \setminus c, A$ meets $\{b_r < c\}$ for all sufficiently large $r$. 
Thus, for such $r$, $b_r \mid_{\Sigma'}$ achieves an inferior minimum $c' < c$ at some point $z \in A$, and hence $\Sigma' \subset \{b_r \geq c'\}$.

- Define the function $\beta_r : I^- (y_r) \to \mathbb{R}$ by

$$
\beta_r(x) = r - \left(\frac{r - c'}{2} + d(x, y_r)\right),
$$

where $y_r = \eta_r \left(\frac{r-c'}{2}\right)$ and $\eta_r : [0, r - c'] \to M$ is maximal geodesic segment from $z$ to $\gamma(r)$.

- Near $z$, the level set $\Sigma_r = \{\beta_r = c'\}$ is smooth spacelike hypers. which meets $\Sigma'$ tangentially at $z$, and lies to the past of $\Sigma'$.

- Therefore, from the Maximum Principle

$$
H_{\Sigma_r}(z) \geq H_{\Sigma'}(z) \geq H/3 > 0. \quad (1)
$$
- On the other hand, from the nonnegative Ricci hypothesis the basic estimate \( \Delta d_q \geq - (n - 1)/d_q \) holds.

- This provides the estimate:

\[
H_{\Sigma_r}(z) < 2(n - 1)/(r - c). \tag{2}
\]

- The contradiction is obtained from (1), (2) by taking \( r \to \infty \). \( \square \)
Nice neighborhoods:

An open set $U \subset I[\gamma]$ is said to be nice (with respect to $\gamma$) if there exist constants $K > 0$ and $T > 0$ such that for each $q \in U$ and $r > T$, any maximal unit speed geodesic segment $\sigma$ from $q$ to $\gamma(r)$ satisfies

$$g_0(\sigma'(0), \sigma'(0)) \leq K,$$

$g_0$ fixed Riemannian metric.
Nice neighborhoods:
An open set $U \subset I[\gamma]$ is said to be nice (with respect to $\gamma$) if there exist constants $K > 0$ and $T > 0$ such that for each $q \in U$ and $r > T$, any maximal unit speed geodesic segment $\sigma$ from $q$ to $\gamma(r)$ satisfies

$$g_0(\sigma'(0), \sigma'(0)) \leq K, \quad g_0 \text{ fixed Riemannian metric.}$$

Properties:
1. For each $t$, $\gamma(t)$ is contained in a nice neighborhood.
2. Asymptotes to $\gamma$ from points in nice neighborhoods are timelike.
3. $\{b_r\}$ converges locally uniformly to $b$ on nice neighborhoods.
   Hence $b$ is continuous on nice neighborhoods.
Level sets of Busemann functions present a nice structure in nice neighborhoods:

**Lemma:**

The level set $\Sigma_c = \{b = c\}$ of a Busemann function $b$ is a partial Cauchy surface at any nice neighborhood $U$.

**Proof:**

- To prove that $\Sigma_c$ is edgeless, assume by contradiction that $p \in \text{edge}(\Sigma_c) \neq \emptyset$.
- There exists a timelike curve in $U$ which goes from $I^-(p, U)$ to $I^+(p, U)$ and does not meet $\Sigma_c$.
- Hence, $b$ does not take the value $c$ along that curve.
- This contradicts the continuity of \( b \), since \( b \) takes values smaller and greater than \( c \) at the extremes of the curve.

- In order to show that \( \Sigma_c \) is acausal, we already know that it is achronal.

- By contradiction, assume that \( \Sigma_c \) is not acausal.

- Then, there exists \( p, q \in \Sigma_c \), \( p \preceq q \), \( p \not\ll q \).

- From Avez-Seifert’s result, there exists a null geodesic \( \eta \) connecting \( p, q \).

- Let \( \{\alpha_n\}_n \) be a sequence of maximal timelike segments connecting \( q \) with \( \gamma(r_n) \) and let \( \alpha \) be a limit timelike geodesic.
- Let $\eta \cdot \alpha_n$ be the resulting curve from cutting the corner to the convolution $\eta \cdot \alpha_n$.

- By making the cuts of the curves appropriately (by comparing them with the corner of $\eta \cdot \alpha$), we deduce

$$d(p, \gamma(r_n)) \geq \text{length}(\eta \cdot \alpha_n) \geq \text{length}(\alpha_n) + \epsilon = d(q, \gamma(r_n)) + \epsilon.$$

- In particular,

$$b_{r_n}(q) - b_{r_n}(p) = d(p, \gamma(r_n)) - d(q, \gamma(r_n)) \geq \epsilon,$$

and so, $b(q) - b(p) \geq \epsilon$, in contradiction with $b(p) = b(q) = c.$ $\Box$
Previous Technical Results

A second convexity result is needed for the proof:

**Lemma:**

Assume $M$ obeys $Ric(v, v) \geq 0$ for all $v$ timelike. Let $\Sigma$ be a connected smooth spacelike hypersurface contained in a “sufficiently small” nice neighborhood of $\gamma(t)$. Assume the mean curvature of $\Sigma$ is nonnegative, $H_\Sigma \geq 0$. If $b$ achieves a minimum along $\Sigma$ then $b$ is constant along $\Sigma$. 

The Splitting Problem in Riemannian and Lorentzian Geometry – p.120/145
A second convexity result is needed for the proof:

**Lemma:**

Assume $M$ obeys $Ric(v, v) \geq 0$ for all $v$ timelike. Let $\Sigma$ be a connected smooth spacelike hypersurface contained in a “sufficiently small” nice neighborhood of $\gamma(t)$. Assume the mean curvature of $\Sigma$ is nonnegative, $H_\Sigma \geq 0$. If $b$ achieves a minimum along $\Sigma$ then $b$ is constant along $\Sigma$.

**Proof:**

- By connectivity, it suffices to prove that $b$ is constant in a neighborhood of the minimum $q$.

- Assume by contradiction that $b$ is not constant in any neighborhood of $q$. 

The Splitting Problem in Riemannian and Lorentzian Geometry – p.121/145
- Let $B$ open coordinate ball $B \subset \Sigma$ centered at $q$ such that $b \mid_{\partial B}$ is not constantly equal to the minimum value.

- Choosing $B$ small enough we can construct a smooth function $h$ on $\Sigma$ such that:
  
  (i) $h(q) = 0$
  
  (ii) $|\nabla_{\Sigma} h| \leq 1$ on $B$, with $\nabla_{\Sigma}$ is the gradient operator on $\Sigma$,
  
  (iii) $\Delta_{\Sigma} h \leq -D$ on $B$, where $D$ is a positive constant and $\Delta_{\Sigma}$ is the induced Laplacian on $\Sigma$, and

  (iv) $h > 0$ on $\partial^0 B = \{x \in \partial B : b(x) = b(q)\}$.

- Define $\beta_{p,r}(x) = r - (l/2 + d(x, y_r))$, where $y_r = \eta_r(l/2)$ and $\eta_r : [0, l] \to M$ is maximal geodesic segment from $p$ to $\gamma(r)$. 

The Splitting Problem in Riemannian and Lorentzian Geometry – p.122/145
- There exists some \( p \in B \), such that the function \( \varphi_{\epsilon,r} := \beta_{p,r} + \epsilon h \) is smooth in a neighborhood of \( p \), and achieves a minimum at \( p \).

- On the other hand, by using the timelike convergence condition, and the mean curvature assumption, it can be proved the following upper estimate:

\[
\Delta_{\Sigma} \varphi_{\epsilon,r}(p) \leq \frac{2(n-1)}{(r-r_0)} + C\epsilon^2 - D\epsilon.
\]

- For \( \epsilon \) small and \( r \) large, one has \( \Delta_{\epsilon} \varphi_{\epsilon,r}(p) < 0 \).

- This contradicts the fact that \( \varphi_{\epsilon,r} \) achieves a minimum at \( p \). \( \square \)
Corollary:

Let $\Sigma$ be a smooth maximal spacelike hypersurface whose closure is contained in a sufficiently small nice neighborhood $U$ of $\gamma(t)$. Assume $\Sigma$ is achronal in $U$ and $\overline{\Sigma}$ is compact. If $\text{edge}(\Sigma) \subset \{b \geq c\}$, then $\Sigma \subset \{b \geq c\}$.

Proof:

- Otherwise, $b$ achieves a minimum value $c' < c$.
- From previous lemma, $b \equiv c'$, in contradiction with $\text{edge}(\Sigma) \subset \{b \geq c\}$. $\square$
Proof of Splitting Theorem
**Sketch of Proof**

**Step 1:** Existence of smooth spacelike hypersurface $\Sigma$ with $b_\pm |_{\Sigma} = 0$.

**Step 2:** There is a line $\alpha$ with $b_\pm (\alpha(t)) = \pm t$, for every point in $\Sigma$.

**Step 3:** Every line $\alpha$ is normal to $B$.

**Step 4:** $E : U = \mathbb{R} \times B \rightarrow E(U)$, $E(t, q) = \exp t N_q$ diffeomorphism.

**Step 5:** $E : U \rightarrow E(U)$ is an isometry (Local Splitting).

**Step 6:** From Local to the Global Splitting.
Proof of Splitting Theorem

Step 1: Existence of smooth spacelike hypersurface $\Sigma$ with $b_\pm |_\Sigma = 0$.

- Let $\gamma$ be the timelike line ensured by the hypotheses of the theorem. By applying the RTI:
  
  $$b_+ + b_- \geq 0 \text{ on } I[\gamma], \quad b_+ + b_- \equiv 0 \text{ on } \gamma.$$

- Denote $S^\pm = \{b_\pm = 0\} \cap U$, with $U$ a nice neighborhood for $\gamma_\pm$.

- From a previous lemma, $S^+$ is a partial Cauchy surface in $U$; hence, it is an embedded topological spacelike hypersurface.

- Let $W$ small coord. ball in $S^+$ centered at $\gamma(0)$, with $\overline{W} \subset S^+$.

- Bartnik’88: $\exists$ solution to the Dirichlet problem for maximal hypersurfaces with contour in a topological spacelike surface.
Proof of Splitting Theorem

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- Let $W$ small coord. ball in $S^+$ centered at $\gamma(0)$, with $\overline{W} \subset S^+$.

- Bartnik’88 $\Rightarrow \exists$ smooth maximal spacelike hypersurface $\Sigma$ which is achronal in $U$, $\overline{\Sigma}$ compact, $\text{edge}(\Sigma) = \text{edge}(W)$ and $\Sigma$ meets $\gamma$. 
In particular, $\text{edge}(\Sigma) \subset \{b_+ \geq 0\} \cap \{b_- \geq 0\}$.

From previous Corollary, $\Sigma \subset \{b_+ \geq 0\} \cap \{b_- \geq 0\}$.

Therefore, $b_\pm = 0$ where $\Sigma$ meets $\gamma$, which must be $\gamma(0)$.

Since $b_+(\gamma(0)) = b_-(\gamma(0)) = 0$, the second convexity lemma implies $b_+ = b_- = 0$ on $\Sigma$. 

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**Proof of Splitting Theorem**

**Step 2:** There is a line $\alpha$ with $b_\pm(\alpha(t)) = \pm t$, for every point in $\Sigma$.

- Let $B \subset \Sigma$ a geodesic ball in $\Sigma$ centered at $\gamma(0)$ of radius $R$.
- From each point of $B$, $\exists$ timelike asymptotes $\alpha_\pm$ to $\gamma_\pm$, resp.
- Let $\alpha : \mathbb{R} \to M$ the (possibly) broken geodesic given by:

\[
\alpha(t) = \begin{cases} 
\alpha_-(t) & -\infty < t \leq 0 \\
\alpha_+(t) & 0 \leq t < \infty. 
\end{cases}
\]

- Since $\alpha_\pm$ are asymptotes to $\gamma_\pm$, and $b_\pm \mid_{\Sigma} = 0$, we have:

\[
b_+(\alpha_+(t)) = b_+(\alpha_+(0)) + t = t \quad \text{if } t \geq 0.
\]

\[
b_-(\alpha_-(t)) = b_-(\alpha_-(0)) + t = t
\]
Proof of Splitting Theorem

- Then:

\[ t = b_+(\alpha_+(t)) \geq b_+(\alpha_-(t)) + d(\alpha_-(t), \alpha_+(t)) \geq b_+(\alpha_-(t)) + 2t, \]

and thus,

\[ b_+(\alpha_-(t)) \leq -t, \quad \text{if } t \geq 0. \]

- Notice also that:

\[ 0 \leq b_+(\alpha_-(t)) + b_-(\alpha_-(t)) \leq -t + t = 0; \]

hence,

\[ b_+(\alpha_-(t)) = -b_-(\alpha_-(t)) = -t, \quad \text{if } t \geq 0. \]
Proof of Splitting Theorem

- Summarizing:

\[ b_+(\alpha(t)) = b_+(\alpha_+(t)) = t \quad \text{if } t \geq 0 \]
\[ b_+(\alpha(t)) = b_+(\alpha_-(t)) = -(t) = t \quad \text{if } t \leq 0. \]

- The expression \( b_-(\alpha(t)) = -t \) is deduced similarly.

- Finally, \( \alpha \) must be a line, since

\[ \text{length}(\alpha \mid_{[t_1,t_2]}) = t_2 - t_1 = b_+(\alpha(t_2)) - b_+(\alpha(t_1)) \geq d(\alpha(t_1), \alpha(t_2)) \]

and thus,

\[ \text{length}(\alpha \mid_{[t_1,t_2]}) = d(\alpha(t_1), \alpha(t_2)). \]
Step 3: Every line $\alpha$ is normal to $B$.

- From inequality $b(q) \geq b(p) + d(p, q)$ we deduce that function
  \[ b_{q,r}^+(x) = r - d(x, \alpha(r)) \]
  is an upper support function of $b_+$ at $q = \alpha(0)$ for $r > 0$.

- Analogously, $b_{q,r}^-(x) = -r + d(\alpha(-r), x)$ is a lower support function of $b_+$ at $q = \alpha(0)$ for $r > 0$.

- Therefore, $b_+$ is differentiable at $q$, and $\nabla b_+(q) = -\dot{\alpha}(0)$.
  (Similar argument proves that $\nabla b_+(\alpha(t)) = -\dot{\alpha}(t)$.)

- Since $\nabla b_+(q)$ is normal to $B$, also is the line $\alpha$. 
Proof of Splitting Theorem

Step 4: $E : U = \mathbb{R} \times B \rightarrow E(U), E(t, q) = \exp tN_q$ diffeomorphism.

$E$ is injective:

- $E$ is injective iff normal geodesics to $B$ do not intersect.
- Future normal geodesics from $B$ are asymptotes to $\gamma_+$. 
- But recall that asymptotes $\alpha$ to $\gamma_+$ satisfy $\nabla b_+(\alpha(t)) = -\dot{\alpha}(t)$. Thus, asymptotes at different points of $B$ do not intersect.
- Similarly, past normal geodesics from $B$ do not intersect.
- Finally, future and past normal geodesics cannot intersect between them because, otherwise, they would violate the achronality of $\Sigma \subset \{b_+ = 0\}$. 

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Proof of Splitting Theorem

$E$ is nonsingular:

- Assume by contradiction that $E$ is singular.

- Let $\alpha(a)$, $a > 0$, be the first focal point to $p \in B$ along $\alpha$.

- $\exists V \subset \mathbb{R} \times B$ of $[0, a) \times \{p\}$ s.t. $E : V \to V'$ is diffeomorphism. In particular, $b_+(\exp tN_q) = t$ on $V$.

- Hence, $b_+$ is smooth on $V$, and $\Delta b_+ = H_{\Sigma_t}$, $\Sigma_t = \{b_+ = t\} \cap V$.

- On the one hand, from the first convexity result we have $H_{\Sigma_t} = \Delta b_+ \leq 0$ along $\alpha \mid_{[0,a]}$.

- On the other hand, since $\alpha(a)$ focal point, $\limsup_{t \to a} H_{\Sigma_t} = \infty$, a contradiction.
Proof of Splitting Theorem

Step 5: \( E : U \rightarrow E(U) \) is an isometry (Local Splitting).

- Recall that \( b_{\pm}(\exp tN_q) = \pm t \). In particular, \( b_{\pm} \) is smooth.
- From the first convexity result, \( \Delta b_{\pm} \leq 0 \) on \( U \).
- This inequality joined to \( b_+ = -b_- \) implies \( \Delta b_+ = 0 \) on \( U \).
- Next, consider the well-known formula:

\[
-\nabla b_+ (\Delta b_+) = \text{Ric}(\nabla b_+, \nabla b_+) + |\text{Hess} b_+|^2, 
\]

\( (\nabla b_+ \) unit past vector field tangent to the normal geodesic to \( B \)).
- Since \( \Delta b_+ = 0 \) and \( \text{Ric}(\nabla b_+, \nabla b_+) \geq 0 \), it is \( \text{Hess} b_+ = 0 \) on \( U \).
- Hence, \( \nabla b_+ \) is parallel on \( U \), and thus, \( E \) is an isometry.
**Proof of Splitting Theorem**

**Step 6: From Local to the Global Splitting.**

* A flat strip is a totally geodesic isometric immersion $f$ of $(\mathbb{R} \times I, -dt^2 + ds^2)$ into $M$ such that $f \mid_{\mathbb{R} \times \{s\}}$ is line $\forall s \in I$.

* Two lines $\gamma_1$, $\gamma_2$ are strongly parallel if they bound a flat strip.

* Two lines $\gamma_1$, $\gamma_2$ are parallel if $\exists$ lines $\gamma_1 = \beta_0, \beta_1, \ldots, \beta_k = \gamma_2$ which are strongly parallel.

- If $\gamma_1$, $\gamma_2$ are parallel lines then $I[\gamma_1] = I[\gamma_2]$ and the Busemann functions $b^\pm_1, b^\pm_2$ agree.

- Denote by $P_\gamma \subset M$ the set of points which lie on a line which is parallel to $\gamma$. 

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Proof of Splitting Theorem

- $b_+$ is differentiable at $P_\gamma$ and there exists exactly one parallel line $\gamma_q$ passing through any $q \in P_\gamma$.

- From the local splitting, we have:
  
  (i) $P_\gamma$ is open.
  
  (ii) $P_\gamma$ is closed.

  (For any geodesic $c : [0, 1] \rightarrow M$ starting from a line $\gamma$, there exists flat strip containing both, $\gamma$ and $c$.)

- Since $M$ is connected, $P_\gamma = M$.

- Therefore, $\exists$ one line $\gamma_q$ parallel to $\gamma$, through every $q \in M$.

- From the local splitting, this defines a parallel timelike vector field $V$ on $M$. 
Proof of Splitting Theorem

- Therefore, $V^\perp$ is a parallel distribution, and so, it is integrable.
- Let $H$ be the maximal integral leave through $p = \gamma(0)$.
- The map

$$j : \mathbb{R} \times H \to M, \quad j(t, q) = \gamma_q(t)$$

is the desired isometry. □
Open Problem
Prototype Singularity Theorem:

Suppose that the spacetime \((M, g)\), of dimension \(n > 2\), satisfies the following conditions:

(1) \((M, g)\) contains a compact Cauchy surface,

(2) \(\text{Ric}(v, v) \geq 0\) for all timelike \(v \in TM\),

(3) certain curvature quantity is nonzero at some point of each inextensible causal geodesic.

Then \((M, g)\) contains an incomplete causal geodesic.

¿Is there some rigidity property associated to this result?
This rigidity question is related to the following idea (Geroch’70):

*Generically, inextensible closed universes satisfying Einstein equations should be timelike geodesically incomplete.*

These and other ideas by Geroch were summarized by Galloway and Horta in 1995 as:

*Spatially closed s-t should fail to be singular only under exceptional circumstances.*

This is connection with the following conjecture formulated by Bartnik:

Suppose that the spacetime \((M, g)\), of dimension \(n > 2\), satisfies the following conditions:

1. \((M, g)\) contains a compact Cauchy surface,
2. \(\text{Ric}(v, v) \geq 0\) for all timelike \(v \in TM\).

Then either \((M, g)\) is timelike geodesically incomplete, or else \((M, g)\) splits isometrically as a product \((\mathbb{R} \times M_1, -dt^2 \oplus g_1)\), where \((M_1, g_1)\) is a compact Riemannian manifold.

Actually, Bartnik also conjectured that \(M\) must contain a constant mean curvature Cauchy surface, but this has been shown to be false!
The Bartnik’s conjecture has been proved under some additional assumptions by:

- Bartnik (1988)
- Ehrlich and Galloway (1990)
- Eschenburg and Galloway (1992)

Recently, Sharifzadeh and Bahrampour (2009) have made contributions to this problem by applying new results about the level sets of Busemann functions for spacetimes.

However, as far as I know, the conjecture remains unsolved in its full generality!
THANK YOU!!