Time Functions

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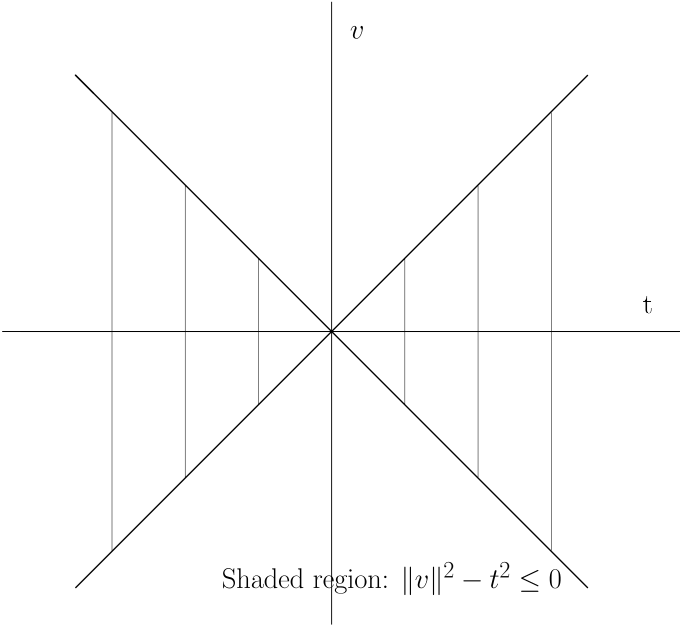
Around 1990 Paul Ehrlich introduced me to the subject.

We start with a Lorentz manifold, i.e. a smooth manifold M of dimension n endowed with a smooth metric $g_x, x \in M$, where the signature of g_x is $(\underbrace{+, \ldots, +}_{n-1}, -)$.

At each point x the set

$$\{v \in T_M \mid g_x(v, v) \leq 0\}$$

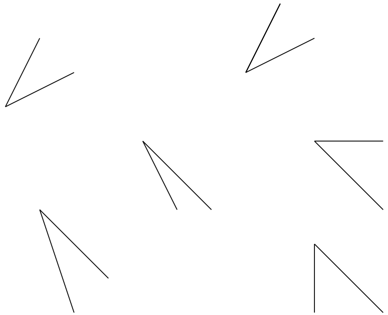
is a double cone with tip at 0.



Shaded region: $\|v\|^2 - t^2 \leq 0$

M is said to be time-oriented if for each $x \in M$ we can choose in a continuous way a cone $\mathcal{C}_x \subset T_x M$, with tip at 0, such that

$$\mathcal{C}_x \cup -\mathcal{C}_x = \{v \in T_M \mid g_x(v, v) \leq 0\}.$$



Choosing $\mathcal{C}_x$

From now on, we assume M time-oriented.

Causal Curves

A causal curve is a C^1 curve $\gamma : [a, b] \to M$ such that $\dot{\gamma}(t) \in \mathcal{C}_{\gamma(t)}$, for every $t \in [a, b]$.

We extend this notion to piecewise C^1 curves and to Lipschitz curves:

- A piecewise C^1 curve $\gamma : [a, b] \to M$ is said to be causal if both right and left derivatives of γ at t are in $\mathcal{C}_{\gamma(t)}$ for every $t \in [a, b]$.
- A Lipschitz curve $\gamma : [a, b] \to M$ is said to be causal if $\dot{\gamma}(t) \in \mathcal{C}_{\gamma(t)}$ for almost every $t \in [a, b]$. Recall that a Lipschitz map has a derivative a.e.

Time Functions

- A smooth time function is a C^∞ function $f : M \to \mathbb{R}$ such that

$$d_x f(v) > 0, \text{ for every } v \in \mathcal{C}_x \setminus \{0\}.$$

- A continuous function $f : M \to \mathbb{R}$ is a continuous time function if $f(\gamma(b)) \geq f(\gamma(a))$ for every causal curve $\gamma : [a, b] \to M$.
- A continuous time function $f : M \to \mathbb{R}$ is said to be strict if $f(\gamma(b)) > f(\gamma(a))$ for every **non-constant** causal curve $\gamma : [a, b] \to M$.

Some very personal view on proofs of existence of time functions on stably causal Lorentzian manifolds

We will explain hereafter what are stably causal Lorentzian manifolds.

- Geroch in the 1960's constructed a strict continuous time function on any stably causal Lorentzian manifold, by a very clever method using volumes of the "past of a point".
- In the 1970's, Seifert published a paper smoothing Geroch's functions, thereby obtaining a smooth time function.
- People believed there was a hole in Seifert's argument.
- This was the situation around 1990, when I learned the problem from Paul Ehrlich, then my colleague at the University of Florida.

- Obviously it remained on my mind. In 2005 with Antonio Siconolfi (University La Sapienza), we realized that we had a proof of existence of smooth time functions for more general cone structures on manifolds as a by-product of our work on Hamilton-Jacobi equations.
- Slightly before Bernal and Sanchez already found a proof of existence of smooth time functions on any stably causal Lorentzian manifold, using new geometrical ideas.
- Recently, in 2012, Chrusciel, Grant & Minguzzi obtained a proof for stably causal Lorentzian manifolds which tends to validate Seifert's idea.

Lyapunov Functions

SORRY ABOUT THAT

Coming from Dynamical Systems, it is more convenient for me to work with Lyapunov functions.
A function $f : M \to \mathbb{R}$ is Lyapunov if $-f$ is a time function, i.e. $f(\gamma(t))$ is a decreasing (rather than increasing) function of t for every causal curve $\gamma : [a, b] \to M$.
Of course, a smooth Lyapunov function is a C^∞ function $f : M \to \mathbb{R}$ such that

$$d_x f(v) < 0, \text{ for every } v \in \mathcal{C}_x \setminus \{0\}.$$

Our method, with Antonio, for constructing smooth time functions works more generally for cone structures on manifolds.

Cone Structure on a Manifold

A cone structure $\mathcal{C}$ on the manifold M, is a family $\mathcal{C} = (\mathcal{C}_x)_{x \in M}$, such that for every $x \in M$:
$\mathcal{C}_x$ is a convex cone in $T_x M$, the tangent space to M at x, with tip at 0, which is also salient (i.e. $\mathcal{C}_x$ contains no line), and whose interior $\mathring{\mathcal{C}}_x$ is not empty.
Moreover, we will assume that $\mathcal{C}_x$ depends continuously on x.

To explain this continuity, endow M with an auxiliary Riemannian metric, and call $\bar{\mathbb{B}}_x$ the unit ball in $T_x M$.
Continuity of the family of cones means that the family $\mathcal{C}_x \cap \bar{\mathbb{B}}_x$ of compact subsets of TM is continuous for the Hausdorff topology on compact subsets of TM.

To make things easier, I will assume from now on $M = \mathbb{R}^n$. In that case $T_x\mathbb{R}^n$ is canonically identified with $\mathbb{R}^n$. We choose as Riemannian metric on $\mathbb{R}^n$ the usual Euclidean metric.
If A, B are compact subsets in $\mathbb{R}^n$

$$\rho(A,B) = \sup_{x \in A} d(x,B), \text{ where } d(x,B) = \inf_{y \in B} \|y - x\|.$$

The Hausdorff distance d_H on the set of compact subsets of $\mathbb{R}^n$ is defined by

$$d_H(A,B) = \max(\rho(A,B), \rho(B,A)).$$

As is well-known $d_H(A,B) \leq \epsilon$ if and only if

$$A \subset \bar{V}_\epsilon(B) \text{ and } B \subset \bar{V}_\epsilon(A),$$

where $\bar{V}_\epsilon(C) = \{x \in \mathbb{R}^n \mid d(x,C) \leq \epsilon\}$ is the closed ϵ-neighborhood of C in the Euclidean space $\mathbb{R}^n$.

The topology on the set on closed convex cones in $\mathbb{R}^n$ with tip at 0 is defined by the distance D given by

$$D(C, C') = d_H(C \cap \bar{B}(0,1), C' \cap \bar{B}(0,1)),$$

where $\bar{B}(0,1) = \{v \in \mathbb{R}^n \mid \|v\| \leq 1\}$ is the unit Euclidean ball in $\mathbb{R}^n$.

A cone structure on $\mathbb{R}^n$ is therefore a family $\mathcal{C} = (\mathcal{C}_x)_{x \in \mathbb{R}^n}$, where $\mathcal{C}_x \subset \mathbb{R}^n$ is a salient convex cone in $\mathbb{R}^n$ whose interior $\mathring{\mathcal{C}}_x$ is not empty, such that the map $x \mapsto \mathcal{C}_x$ is continuous.

The notion of causal curve makes perfect sense for cone structures: A C^1 (resp. Lispschitz) curve $\gamma : [a, b] \to M$ is causal (for $\mathcal{C}$) if $\dot{\gamma}(t) \in \mathcal{C}_{\gamma(t)}$, for every (resp. almost every) $t \in [a, b]$.

Since the notion of (smooth) Lyapunov functions involves only causal curves and cones, they also make perfect sense for a cone structure.

Causal family of cones

Recall that a loop is a curve $\gamma : [a, b] \to \mathbb{R}^n$ with $\gamma(a) = \gamma(b)$.

If there is a continuous strict Lyapunov function for $\mathcal{C}$, then any closed causal loop $\gamma : [a, b] \to \mathbb{R}^n$ must be constant. In fact, if there existed $t \in]a, b[$, with $\gamma(t) \neq \gamma(a) = \gamma(b)$ applying the strict condition to the causal curves $\gamma|[a, t]$ and $\gamma|[t, b]$, we would have

$$f(\gamma(a)) > f(\gamma(t)) > f(\gamma(b)).$$

This would contradict the equality $\gamma(a) = \gamma(b)$.

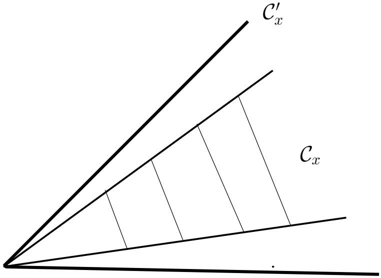
This leads to the (known) definition of causal cone structure.

A cone structure is causal if it has no non-constant closed causal loop.

Enlargement

In fact, the existence of a smooth Lyapunov function adds more restriction.
To explain this we first introduce the notion of enlargement.
A cone structure $\mathcal{C}'$ is an enlargement of the cone structure $\mathcal{C}$ if we have

$$\mathring{\mathcal{C}}'_x \supset \mathcal{C}_x \setminus \{0\}, \text{ for every } x \in \mathbb{R}^n.$$



If f is a smooth Lyapunov function, then

$$\eta(x) = \sup\{d_x f(v) \mid v \in \mathcal{C}_x, \|v\| = 1\},$$

is < 0 everywhere and continuous on $\mathbb{R}^n$. We set

$$\mathcal{C}'_x = \{v \mid d_x f(v) \leq \frac{\eta(x)}{2}\|v\|\}.$$

Using $\eta < 0$, we see that $\mathcal{C}' = (\mathcal{C}'_x)_{x\in\mathbb{R}^n}$ is a cone structure, which is an enlargement of $\mathcal{C}$. Obviously f is a smooth Lyapunov function for $\mathcal{C}'$. Hence this enlargement $\mathcal{C}'$ is causal.

This leads to the definition:
A cone structure is stably causal if it has a causal enlargement.

Stable causality is therefore a necessary condition for the existence of a smooth Lyapunov function.

The converse is true.

Theorem

Any stably causal cone structure admits smooth Lyapunov (or time) functions

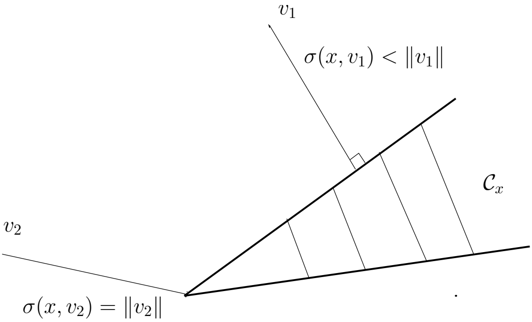
In the remaining part of the lecture we sketch how to construct strict locally Lipschitz Lyapunov functions.

Although smoothing a strict locally Lipschitz Lyapunov function to a smooth Lyapunov function requires some care, the process is quite well understood now,
Our work with Antonio Siconolfi, on smoothing weak solutions of the Hamilton-Jacobi equation, provides a flexible method to do the smoothing process, using convolution and partitions of unity.

To construct a locally Lipschitz strict Lyapunov function, we introduce, at each $x \in \mathbb{R}^n$, a "line element" $\sigma_{\mathcal{C}}(x, \cdot)$ adapted to our problem, defined by:

$$\sigma_{\mathcal{C}}(x, v) = d(v, \mathcal{C}_x) = \inf_{c \in \mathcal{C}_x} \|v - c\| \leq \|v\|,$$

where the last inequality follows from the fact that $0 \in \mathcal{C}_x$.



Since $\mathcal{C}$ is fixed, we will write $\sigma(x, v)$ instead of $\sigma_{\mathcal{C}}(x, v)$.

For x fixed $\sigma(x,v)$ is ≥ 0, and convex as an inf of convex functions.
Since $\mathcal{C}_x$ is a cone with tip at 0, it is also positively homogeneous (i.e. $\sigma(x,tv) = t\sigma(x,v)$, for $t \geq 0$).
But a positively homogeneous convex function is subadditive, therefore we have

- $0 \leq \sigma(x,v) \leq \|v\|$, and $\sigma(x,v) = 0$, if and only if $x \in \mathcal{C}$,
- $\sigma(x, v_1 + v_2) \leq \sigma(x, v_1) + \sigma(x, v_2)$,
- $\sigma(x,tv) = t\sigma(x,v)$.

Moreover, by the continuity of $x \mapsto \mathcal{C}_x$, the function $(x,v) \mapsto \sigma(x,v)$ is also continuous.
This means that σ is a non-symmetric degenerate Finsler metric.

The σ-length ℓ_σ of curves

Like for every Finsler metric we can define the notion of the σ-length $\ell_\sigma(\gamma)$ of the Lipschitz curve $\gamma : [a, b] \to \mathbb{R}^n$ by

$$\ell_\sigma(\gamma) = \int_a^b \sigma(\gamma(s), \dot\gamma(s))\, ds.$$

Since $\sigma(x, v) \leq \|v\|$, we have

$$\ell_\sigma(\gamma) \leq \int_a^b \|\dot\gamma(s)\|\, ds = \ell_{\mathrm{euc}}(\gamma),$$

where $\ell_{\mathrm{euc}}(\gamma)$ is the Euclidean length of γ.
Moreover, since $\sigma(x, v) = 0$ for $v \in \mathcal{C}_x$, we get

$$\ell_\sigma(\gamma) = 0, \text{ for every causal curve } \gamma.$$

The σ-distance S_σ

Once we have a length for curves, we can define a distance S_σ on $\mathbb{R}^n$ by

$$S_\sigma(x,y) = \inf\{\ell_\sigma(\gamma) \mid \gamma : [a,b] \to \mathbb{R}^n, \gamma(a) = x, \gamma(b) = y\}.$$

The properties of S_σ are

1) $S_\sigma(x,x) = 0$, since a constant path has σ-length equal to 0.

2) $0 \le S_\sigma(x,y) \le \|y - x\|$, since $0 \le \ell_\sigma \le \ell_{\mathrm{euc}}$.

3) $S_\sigma(x,z) \le S_\sigma(x,y) + S_\sigma(y,z)$, by concatenation of paths.

This implies that

$$\begin{aligned} |S_\sigma(x',y') - S_\sigma(x,y)| &\le S_\sigma(x,x') + S_\sigma(y,y') \\ &\le \|x' - x\| + \|y' - y\|. \end{aligned}$$

Hence
4) S_σ is Lipschitz on $\mathbb{R}^n \times \mathbb{R}^n$.

5) If $\gamma : [a, b] \to \mathbb{R}^n$ is a causal curve then $S_\sigma(\gamma(a), \gamma(b)) = 0$, since

$$0 \leq S_\sigma(\gamma(a), \gamma(b)) \leq \ell_\sigma(\gamma) = 0.$$

This implies that

$$S_\sigma(x, \gamma(b)) \leq S_\sigma(x, \gamma(a)) + S_\sigma(\gamma(a), \gamma(b)) = S_\sigma(x, \gamma(a)).$$

Therefore
6) For every $x \in \mathbb{R}^n$, the function $y \mapsto S_\sigma(x, y)$ is a Lipschitz Lyapunov function.

Conformal change

Instead of using the line element σ, we could have used $\alpha\sigma$, where $\alpha : \mathbb{R}^n \to]0, +\infty[$ is a continuous function.
This would give the $\alpha\sigma$-length of curves $\ell_{\alpha\sigma}$, and the $\alpha\sigma$-distance $S_{\alpha\sigma}$ defined by

$$\ell_{\alpha\sigma}(\gamma) = \int_a^b \sigma(\gamma(s), \dot\gamma(s))\, ds,$$

$$S_{\alpha\sigma}(x, y) = \inf\{\ell_\sigma(\gamma) \mid \gamma : [a, b] \to \mathbb{R}^n, \gamma(a) = x, \gamma(b) = y\}.$$

Like for properties 1) 3) and 5) of S_σ, it is easy to see that we have
1') $S_{\alpha\sigma}(x, x) = 0.$
3') $S_{\alpha\sigma}(x, z) \leq S_{\alpha\sigma}(x, y) + S_{\alpha\sigma}(y, z)$
5') If $\gamma : [a, b] \to \mathbb{R}^n$ is a causal curve then $S_{\alpha\sigma}(\gamma(a), \gamma(b)) = 0$

Property 2) gets replaced by
2') if D is a convex domain in $\mathbb{R}^n$, then

$$0 \le S_{\alpha\sigma}(x,y) \le K(D)\|y-x\|, \text{ for every } x,y \in D$$

where $K(D) = \sup_{z\in D} \alpha(z)$.

Therefore 4) and 6) get replaced by

4') $S_{\alpha\sigma}$ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^n$.
6') For every $x \in \mathbb{R}^n$, the function $y \mapsto S_{\alpha\sigma}(x,y)$ is a locally Lipschitz Lyapunov function.

Of course these functions $S_{\alpha\sigma}(x,.)$ are not necessarily strict Lyapunov.
Note that we have not yet used the stable causality.

In fact it is much better to take into account all the $S_{\alpha\sigma}(x, \cdot)$, for $x \in \mathbb{R}^n$.
A way to do this is to fix a dense sequence $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^n$. It is not difficult to show that we can then choose a sequence $\eta_i > 0, i \in \mathbb{N}$ such that

$$\theta(y) = \sum_{i\in\mathbb{N}} \eta_i S_\sigma(x_i, y)$$

converges uniformly and is Lipschitz on each compact subset of $\mathbb{R}^n$.
Fix a causal curve $\gamma : [a, b] \to \mathbb{R}^n$. For every $i \in \mathbb{N}$, we have

$$S_{\alpha\sigma}(x_i, \gamma(a)) \geq S_{\alpha\sigma}(x_i, \gamma(b))$$

therefore $\theta(\gamma(a)) \geq \theta(\gamma(b))$, and θ is a locally Lipschitz Lyapunov function

We now would like to see what can prevent the function θ from being a strict Lyapunov function. For this, fix a causal curve $\gamma : [a, b] \to \mathbb{R}^n$, with $\gamma(a) \neq \gamma(b)$, and we assume $\theta(\gamma(a)) = \theta(\gamma(b))$.

Since $S_{\alpha\sigma}(x_i, \gamma(a)) \geq S_{\alpha\sigma}(x_i, \gamma(b))$, and $\eta_i > 0$, for every $i \in \mathbb{N}$, using $\theta = \sum_{i \in \mathbb{N}} \eta_i S_\sigma(x_i, \cdot)$, we obtain

$$S_{\alpha\sigma}(x_i, \gamma(a)) = S_{\alpha\sigma}(x_i, \gamma(b)), \text{ for every } i \in \mathbb{N},$$

By density of the x_i's and continuity of $S_{\alpha\sigma}$, we obtain

$$S_{\alpha\sigma}(x, \gamma(a)) = S_{\alpha\sigma}(x, \gamma(b)), \text{ for every } x \in \mathbb{R}.$$

In particular, setting $x = \gamma(b)$, we get

$$S_{\alpha\sigma}(\gamma(b), \gamma(a)) = S_{\alpha\sigma}(\gamma(b), \gamma(b)) = 0$$

By definition of $S_{\alpha\sigma}$, the equality $S_{\alpha\sigma}(\gamma(b), \gamma(a)) = 0$ provides a sequence of curves $\gamma : [b, b_m] \to \mathbb{R}^n$, $m \in \mathbb{N}$ with $\gamma_m(b) = \gamma(b)$, $\gamma_m(b_m) = \gamma(a)$, and $\ell_{\alpha\sigma}(\gamma_m) \to 0$, as $m \to \infty$.

Since $\ell_{\alpha\sigma}(\gamma) = 0$, we get a sequence of loops $\gamma_m \star \gamma$ based at $\gamma(a)$ with $\ell_{\alpha\sigma}(\gamma_m \star \gamma) \to 0$. On the other hand

$$\ell_{\mathrm{euc}}(\gamma_m \star \gamma) \geq \ell_{\mathrm{euc}}(\gamma) \geq \|\gamma(b) - \gamma(a)\| > 0.$$

This suggests to introduce the **Aubry set** $\mathcal{A}(\alpha\sigma)$ as the set of points $x \in \mathbb{R}^n$ for which we can find a sequence of loops δ_m based at x with $\ell_{\alpha\sigma}(\delta_m) \to 0$, as $m \to \infty$, and $\inf_m \ell_{\mathrm{euc}}(\delta_m) > 0$.
The analysis above shows:

Proposition

If the Aubry set $\mathcal{A}(\alpha\sigma)$ is empty for some continuous function $\alpha : \mathbb{R}^n \to]0, +\infty[$, then there exists a strict locally Lipschitz Lyapunov function.

Therefore the existence of strict locally Lipschitz Lyapunov functions follows from:

Lemma

If the cone structure $\mathcal{C}$ is stably causal, then there exists a continuous function $\alpha : \mathbb{R}^n \to]0, +\infty[$ with $\mathcal{A}(\alpha\sigma) = \emptyset$.

In fact what is shown is that for a given enlargement $\mathcal{C}'$ of $\mathcal{C}$, we can find a continuous function $\alpha : \mathbb{R}^n \to]0, +\infty[$ such that through every point of $\mathcal{A}(\alpha\sigma)$ passes a $\mathcal{C}'$-causal loop.

To conclude this presentation, we would like to show that the Aubry set is not an artifact of the proof but a necessity.

Lemma

If $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth Lyapunov function then there exists a continuous function $\alpha : \mathbb{R}^n \to]0, +\infty[$ such that $\mathcal{A}(\alpha\sigma) = \emptyset$.

To prove this lemma, we suppose that such an f exists.
If $v \in \mathbb{R}^n$ and $c \in \mathcal{C}_x$, using $d_x f \leq 0$ on $\mathcal{C}_x$, we get

$$d_x f(v) \leq d_x f(v) - d_x f(c) \leq \|d_x f\| \|v - c\|,$$

the last inequality coming from the definition of the norm of a linear map.
If we take the inf on $c \in \mathcal{C}_x$, we obtain

$$d_x f(v) \leq \|d_x f\| \sigma(x, v).$$

We now set $\alpha(x) = \|d_x f\| + 1$. Obviously α is > 0 and continuous.

We define Σ by

$$\Sigma(x, v) = \alpha(x)\sigma(x, v) - d_x f(v).$$

Σ is continuous in (x, v) and positively homogeneous in v.

Claim $\Sigma \geq 0$, and $\Sigma(x, v) = 0$ if and only if $v = 0$.

$$\begin{aligned}\Sigma(x, v) &= \alpha(x)\sigma(x, v) - d_x f(v) \\ &= (\|d_x f\| + 1)\sigma(x, v) - d_x f(v) \\ &= (\|d_x f\|\sigma(x, v) - d_x f(v)) + \sigma(x, v)\end{aligned}$$

Note that each one of the 2 terms in the sum is ≥ 0, since we have shown that $d_x f(v) \leq \|d_x f\|\sigma(x, v)$. Therefore $\Sigma \geq 0$.

Moreover, if $\Sigma(x, v) = 0$ then $\|d_x f\|\sigma(x, v) - d_x f(v) = 0$ and $\sigma(x, v) = 0$. Hence $v \in \mathcal{C}_x$, and $d_x f(v) = 0$. Since f is a smooth Lyapunov function, this implies $v = 0$.

It follows from the claim that $\epsilon(x) = \inf\{\Sigma(x, v) \mid \|v\| = 1\}$ is > 0 everywhere.
The function ϵ is also continuous by the continuity of Σ.
By homogeneity of Σ in v, and the definitions of Σ and ϵ, we obtain

$$\alpha(x)\sigma(x, v) - d_x f(v) \geq \epsilon(x)\|v\|.$$

We now show that necessarily $\mathcal{A}(\alpha\sigma) = \emptyset$.
We argue by contradiction.
Assume $x_0 \in \mathcal{A}(\alpha\sigma)$. We can find a sequence of $\delta_m : [a_m, b_m] \to \mathbb{R}^n$ is a sequence of loops based at x_0 with $\ell_{\alpha\sigma}(\delta_m) \to 0$, as $m \to \infty$, and $r = \inf_m \ell_{\mathrm{euc}}(\delta_m) > 0$.

Since $\ell_{\mathrm{euc}}(\delta_m) \geq r$, we can find $c_m \in [a_m, b_m]$ with $\ell_{\mathrm{euc}}(\delta_m|[a_m, c_m]) = r$.
In particular $\delta_m([a_m, c_m]) \subset \bar{B}(x_0, r)$.
If we set $\epsilon_0 = \inf_{\bar{B}(x_0,r)} \epsilon$, the compactness of $\bar{B}(x_0, r)$, and the continuity of ϵ imply $\epsilon_0 > 0$.

Since δ_m is a loop, we have

$$\int_{a_m}^{b_m} d_{\delta_m(s)} f(\dot{\delta}_m(s))\, ds = f(\delta(b_m)) - f(\delta(a_m)) = 0.$$

Therefore

$$\begin{aligned}\ell_{\alpha\sigma}(\delta_m) &= \int_{a_m}^{b_m} \alpha(\delta_m(s))\sigma(\delta_m(s), \dot{\delta}_m(s))\, ds \\ &= \int_{a_m}^{b_m} \alpha(\delta_m(s))\sigma(\delta_m(s), \dot{\delta}_m(s)) - d_{\delta_m(s)} f(\dot{\delta}_m(s))\, ds.\end{aligned}$$

Using $\alpha(x)\sigma(x, v) - d_x f(v) \geq \epsilon(x)\|v\| \geq 0$ everywhere, we get

$$\begin{aligned}\ell_{\alpha\sigma}(\delta_m) &\geq \int_{a_m}^{c_m} \alpha(\delta_m(s))\sigma(\delta_m(s), \dot{\delta}_m(s)) - d_{\delta_m(s)} f(\dot{\delta}_m(s))\, ds \\ &\geq \int_{a_m}^{c_m} \epsilon(\delta_m(s))\|\dot{\delta}_m(s)\|\, ds.\end{aligned}$$

Since $\gamma_m([a_m, c_m]) \subset \bar{B}(x_0, r)$, $\epsilon_0 = \inf_{\bar{B}(x_0,r)} \epsilon > 0$, and $\ell_{\mathrm{euc}}(\gamma_m|[a_m, c_m]) = r$, we obtain

$$\begin{aligned} \ell_{\alpha\sigma}(\delta_m) &\geq \int_{a_m}^{c_m} \epsilon(\delta_m(s)) \|\dot{\delta}_m(s)\| \, ds \\ &\geq \int_{a_m}^{c_m} \epsilon_0 \|\dot{\delta}_m(s)\| \, ds \\ &= \epsilon_0 \ell_{\mathrm{euc}}(\delta_m|[a_m, c_m]) = \epsilon_0 r > 0. \end{aligned}$$

This obviously contradicts the condition $\ell_{\alpha\sigma}(\delta_m) \to 0$.