Variational Tools and Geodesics

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8.9.2014 Preliminaries on critical points and variational principles

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10.9.2014 An intrinsic approach: global hyperbolicity and Killing vector fields

Introduction

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- Johann Bernoulli (1667–1748) The *brachystochrone problem*: *"Given two points A and B in a vertical plane, to find for the movable particle M the path AMB, descending along which by its own gravity, and beginning to be urged from the point A, it may in the shortest time reach the point B"* (1697).

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- Joseph Louis Lagrange (1736–1813) The *Euler–Lagrange equations* (in 1750s).

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where $\mathcal{L} : (t, x, v) \in [a, b] \times M \times TM \mapsto \mathcal{L}(t, x, v) \in \mathbb{R}$, the Lagrangian function, is a real–valued function with continuous first partial derivatives (here, M manifold with TM tangent bundle);

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x_a, $x_b \in M$ are two given points.

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i.e., $x \in \Omega_{a,b}$ is such that

$$dJ(x) = 0.$$

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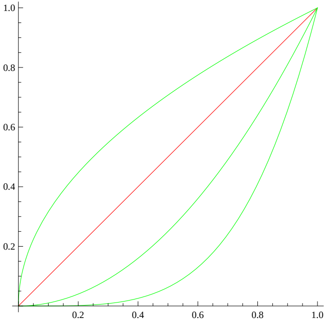
New setting: Fix two points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ in the xy plane, with $x_1 < x_2$, and consider a smooth curve such that

$$\begin{cases} x = t, \\ y = y(t) \end{cases}, \quad t \in [x_1, x_2], \qquad \text{with } y(x_1) = y_1,\ y(x_2) = y_2.$$

Then, the length of this arc is given by

$$J(y) = \int_{x_1}^{x_2} \sqrt{1 + [y'(x)]^2} dx.$$

The problem becomes choosing the function $y = y(x)$ so that the functional J has the smallest possible value.


1.0
0.8
0.6
0.4
0.2
0.2
0.4
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1.0

Geodesic problems

Second problem (Riemannian sphere): taking two points on the surface of a sphere, which arc, lying on the surface and connecting such two points, has the shortest possible length?

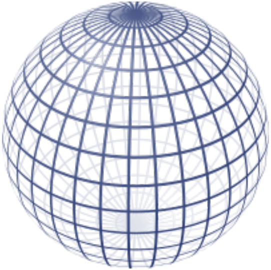

Figure: Riemannian sphere

Variational approach: the minimum

In the previous cited problems common necessity is to minimize a quantity such as time, resistence, length, energy, action, ... (it depends on the model problem), which can be described by a functional defined in a suitable set.

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Abstract setting
Let $J : \Omega \to \mathbb{R}$ be a (given) functional defined on a (suitable) manifold Ω such that

$$\inf_{x \in \Omega} J(x) \; > \; -\infty.$$

The interest is to find sufficient conditions in order to prove that J attains its infimum in Ω.

Variational approach: critical points

A functional equation

$$F(x) = 0,$$

for x belonging to some Banach space B (in particular, a Banach manifold), has *variational form* if there exists a functional $J : B \to \mathbb{R}$ of class C^1 such that the given equation reduces to the Euler–Lagrange equation

$$dJ(x) = 0 \quad \text{in } B. \tag{1}$$

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The true problem is solving (1), i.e., finding critical levels of J in Ω, where Ω is a suitable set of functions. Thus, studying the existence of a minimum point is just a particular case of such a more general investigation.

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if $1 \leq p < +\infty$,

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- Taking $u \in L^1_{loc}(\Omega)$, a function $v_\alpha \in L^1_{loc}(\Omega)$ is the *weak or distributional partial derivative* of u if

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- Sobolev spaces with $\Omega \subseteq \mathbb{R}^N$ $(N \geq 1)$, $m \in \mathbb{N}$, $1 \leq p \leq +\infty$:

$$W^{m,p}(\Omega) \ = \ \{u \in L^p(\Omega) : \ D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\},$$

$$\|u\|_{m,p} := \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_p^p \right)^{1/p} \qquad \text{if } 1 \leq p < +\infty,$$

$$\|u\|_{m,\infty} := \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_\infty \quad \text{if } p = +\infty,$$

$$W_0^{m,p}(\Omega) \ \equiv \ \text{the closure of } C_0^\infty(\Omega) \text{ in } W^{m,p}(\Omega),$$

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◇ B. O'Neill, *Semi–Riemannian Geometry with applications to General Relativity*, Academic Press, New–York, 1983

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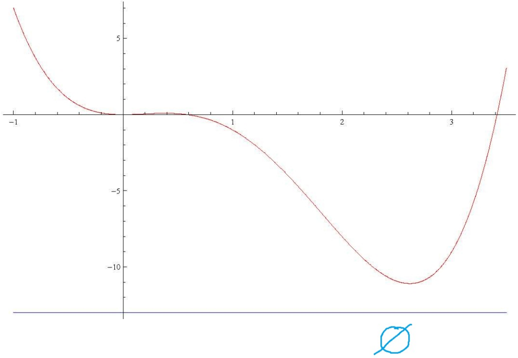
Example

Taking the real function $J(x) = x^4 - 4x^3 + 2x^2$ and a real value $a \in \mathbb{R}$ the corresponding sublevel is

$$J^a = \{x \in B : J(x) \le a\}.$$

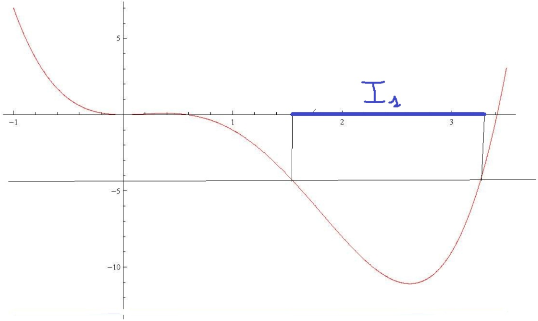
$J(x) = x^4 - 4x^3 + 2x^2, \; a < \min J(\mathbb{R})$

$J^a = \emptyset$



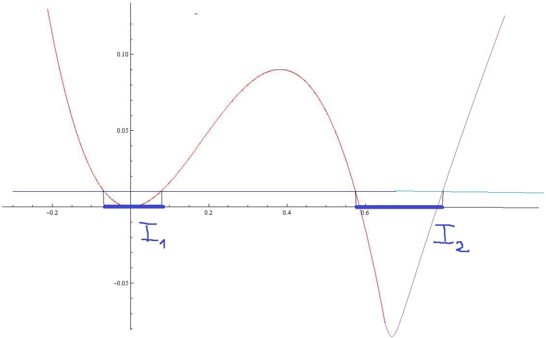
$J(x) = x^4 - 4x^3 + 2x^2$, $\min J(\mathbb{R}) < a < 0$

$J^a = I_1$



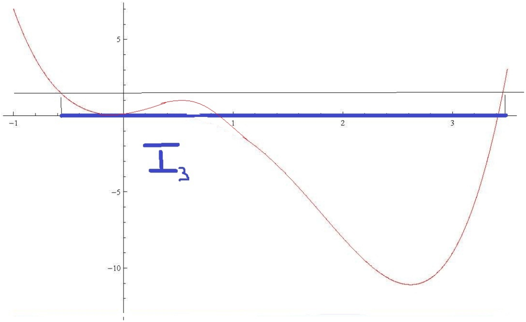
$J(x) = x^4 - 4x^3 + 2x^2$, $0 < a < J(\frac{3-\sqrt{5}}{2})$

$$J^a = I_1 \cup I_2$$



$J(x) = x^4 - 4x^3 + 2x^2,\ a > J(\frac{3-\sqrt{5}}{2})$

$J^a = I_3$



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