A short course on Quantum Mechanics and its Geometry

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Abstract. These notes collect the lectures for the pre-mini course given by the author
to the ”XXIII International Workshop on Geometry and Physics” held in Granada
(Spain) in September 2014.

The aim of these lecture is first to introduce a mathematical oriented reader to
the real of Quantum Mechanics and then to present the geometric structures that
underline the mathematical formalism of Quantum Mechanics which, contrary to what
is usually done in classical physics, are usually not taught in introductory courses. The
mathematics related to Hilbert spaces and Differential Geoemtry are assumed to be
known by the reader.
(Preliminary version: some additional content and references will be added
subsequently.)

Introduction

Whenever teaching introductory courses in either classical or quantum physics, one is
lead to answer at first the following question: what are the minimal conceptual and
mathematical structures required for the description of a physical system?

According to modern knowledge, we essentially need some main ingredients that can be
identified with:

• a space of states $S$, a state being an object which is able to encode all degrees of
freedom of the system;
• a space of observables $O$, which can be applied to a state to change it in some
specified manner;
• a pairing $\mu : S \times O \to \mathbb{R}$, which produces a real number out of a state and an
observable and corresponds to a measurement process;
• a law of evolution, which fixes (possibly in a unique way) how states and/or
observables changes as we let the system evolve, eventually under the influence
of external and internal forces.
In Classical Mechanics (CM): $S$ is the collections of the independent coordinates and momenta $\{q_i, p_i\}$ describing the system, mathematically $S$ is a symplectic manifold; $O$ is given by (usually differentiable) real functions on $S$; the process of measure corresponds to "calculate" an observable on a state, more precisely $\mu$ is a probability measures; while the evolutions laws are given in terms of differential equations, which can be derived as a special observable, the Hamiltonian, is specified.

In these lectures for the pre-mini course, we would like to give an introduction to the analogous set-up necessary to describe Quantum Mechanics (QM). In particular, in the first part, we are interesting in showing how the phenomena about matter and radiation discovered starting from the beginning of the XX century forced physicists and mathematical physicists to develop a whole new set of ideas. We have no intention to be faithful to the historical and non-linear developments that accompanied these discoveries. In what follows we will concentrate on some aspects that we believe to be essential for the understanding the framework of the new quantum mechanical theory. In the second part, we we will concentrate on the mathematical aspects and in particular we will analyse which are the geometric objects that give the underlying structure of QM.

Since the very beginning, the conceptual foundations of QM have been the subject of an intense debate, that goes beyond the scope of these lectures. In the following course, we will try to give some hints only one problem, which is however of primary importance: that about "Methods of Quantization". Indeed, if what the so-called classical limit or dequantization process is to some extent quite well understood, there exists no univocal prescriptions to "quantize" a classical system, i.e. to give a set of unambiguous rules that allow to write the QM version of a given classical system.

These lectures are organised as follows.

Sect. 1 describes the main phenomenology that led to the set-up of QM and introduces the conceptual and mathematical "postulates" that describe the theory. In particular, we will describe the notions of space of states (a Hilbert space $\mathcal{H}$) and of observables (self-adjoint operators $O$), and of evolution (Schroedinger equation). We will look also at some key examples. Despite being of extreme interest, we will not discuss the concept of measurement process, since this topic will take us too far from our main aim. We will end this part with a brief introduction to the so-called GNS construction which yields a duality relation between the concepts of states and observables.

Most of the work done in this section is at the level of Hilbert spaces, where the linearity principle is enforced. However, the physical content of a state is encoded in a vector of the Hilbert space up to multiplications of a (non-zero) complex number. Thus the physical space of state is not $\mathcal{H}$ itself, by the projective Hilbert space $P\mathcal{H}$ which, despite not being a vector space anymore, has a rich geometric structure that will be

* This holds true as far the concepts of states, observables and evolution concern. As for the measurement process, the problem, connected to the question of decoherence, is still open.
investigated in Sect. 2. More specifically we will show how the hermitean form on $\mathcal{H}$, projects down to $P\mathcal{H}$, making it a Kähler manifold.

1. Birth of Quantum Mechanics

1.1. The need of a new theory

The XIX century was the apex of Classical Mechanics. The newly born tools of differential and integral calculus and new theoretical general principles (such as variational ones) allowed to put on a rigorous basis what we now call Analytical Mechanics and provided the framework to study all mechanical problems, from the simple case of a single point particle, to planetary motion, to rigid bodies. Also, the study of electric and magnetic forces culminated in the work of J.C. Maxwell, which not only provided a unification of these two originally different phenomena using the concept of electromagnetic field, but also unified electromagnetism with the theory of light, via the notion of electromagnetic waves.

Thus at the beginning of the XX century, physicists (and mathematical physicists) essentially worked with two paradigms, according to which they could study all known phenomena*:

- matter: described by corpuscles, denumerable and localised: position and momentum at a given instant of time are the quantities defying their motion, knowing which one can calculate any other observable, such as energy (once the mass distribution is given);
- fields: described by waves, continuous and delocalised: to study their motion one needs the concept of wavelength (or frequency), propagation speed and amplitude of the oscillation.

Let me underline that, despite the discreteness implied in the notion of corpuscle, in both these cases, observables, such as energy, are continuous variables whose time evolution is fixed by differential equations (Hamilton eq. and Maxwell eq. respectively), that are second order in time.

From the end of the XIX century, more and more compelling experimental evidences started questioning the great success of CM and its paradigm. The new physics emerged when people began to study the interaction of light with matter and matter itself at a microscopic level. To recall all these facts goes beyond the scope of these lectures and a discussion of them can be found in most introductory books in QM (see e.g. [5, 10]). In the following, we will go over only some fundamental points to understand the change of perspective needed to go from classical to quantum theory, without thinking of being exhaustive and giving a complete picture.

* Of course there are points of contacts between these two approaches. There are situations in which point-like particles originate collective motions that can be interpreted as waves, such as in fluidodynamics. Also, geometric optics and the corpuscular behaviour of light can be obtained as a suitable limit of wave theory. We will not discuss these topics here.
(i) **Blackbody Radiation.** The spectrum of emission/absorption of a black body could not be explained through classical electromagnetism theory, which leads to an infinite amount of emitted/absorbed energy (this is known as "ultra-violet catastrophe"). The problem was solved by Planck in 1900, by introducing the concept of a *quantum* of emitted/absorbed radiation. In his seminal paper [17], he introduced for the first time the idea that the exchanged energy was not continuous but had to be fixed by the frequency $\nu$ of the radiation:

$$E = h\nu$$  \hspace{1cm} (1)

where $h$ is a fundamental constant, now called Planck constant. Notice that $h$ has the dimension of an action: $[h] = LM^2T^{-1}$.

(ii) **Photoelectric effect.** In one of its 1905 paper [7], to describe among other things the absorption of light by electrons in a metal, Einstein introduced the idea that the electromagnetic radiation would propagate as *quanta* carrying a discrete quantity of energy, proportional to the frequency of the radiation, as in* eq. (1).

(iii) **Atomic spectra.** In 1916, in order to solve the problem of stability of atoms and the discrete line shape of atomic spectra, N. Bohr introduces [2] the concept that electrons revolving about the (positively charged) nucleus could not occupy any orbit (as for a planetary system), but those with a discretised value of angular momentum and energy:

$$L_z = m\hbar , \ m \in \mathbb{N}$$  \hspace{1cm} (2)

$$E = E_0/n^2 , \ n = \{1, 2, \ldots\}$$  \hspace{1cm} (3)

where $\hbar = h/2\pi$ and $E_0$ represents the energy of the lowest (ground) admissible level.

(iv) **De Broglie relation.** In 1924, De Broglie proposed that, in the same way as electromagnetic waves can be described by discretised corpuscles (i.e. quanta of light, later denominated photons), particles composing matter may be described as a wave, whose wavelength is connected to the momentum of the particle via the famous relation:

$$\lambda = h/p$$  \hspace{1cm} (4)

These conceptually new descriptions of radiation and matter, which attributes a corpuscular behaviour to electromagnetic waves and an ondulatory behaviour to fundamental particles, are at the origin of the so-called *particle-wave duality*, that reigns the quantum world. The nature and the meaning of this principle has been discussed inside the physics as well the philosophy community since its formulation. Without entering in such a discussion, let me stress that its assumption forces us to recognise that the quantum world has to be described by means of a new physical theory, accompanied by a suitable new mathematics, in which classical paradigma are

*In passing, we notice that Einstein did not recognised immediately that its constant of proportionality was the same introduced by Planck to explain blackbody radiation.*
no longer valid: quantum objects are neither particles nor waves, even if they might look as if behaving like a particle or like a wave in some circumstances, and they have to be described in terms of a new set of principles [12].

It is possible to recognise the consequences of this duality principle all over the development of quantum theory. In the following I will describe some experimental cases in which it can be immediately invoked. Further on, we will see it "in action" when we will present the so-called Schroedinger and Heisenberg approach to QM, but its effects can be seen up to more recent conceptual developments such as the definition of a quantum field (both matter and interaction field) and the technique of second quantization.

A first application of the De Broglie relation, allows us to re-interpret Bohr’s quantization rule for the hydrogen atom. The latter is a particular application of the so-called Bohr-Sommerfeld condition, which represents a first attempt to "quantize" a system. Given a dynamical system with $N$ degrees of freedom, if we denote with $q^1, \cdots, q^N$ the position coordinates and with $p^1, \cdots, p^N$ the corresponding momenta, the "quantization condition" reads as:

$$\oint p_j dq_j = n_j h, \quad n_j \in \mathbb{N}, \quad j = 1, \cdots, N$$

(5)

where the integral is done over a loop. In particular, if we consider a circular orbit of length $\ell$ described by an angular position $q = \phi$ and take the associated (angular) momentum $p = \cos \phi$, the above equations yields:

$$p\ell = nh$$

(6)

If we put this together with (4), we see that Bohr’s quantization condition can be interpreted as saying that the allowed orbits are those which contain an integer number of wavelengths: $\lambda = \ell/n$.

As a more direct proof’s of De Broglie relation, one can look for wave-light behaviour of matter. Indeed one can immediately infer that a beam of particles, say electrons, should exhibits phenomena that are typical of waves, such as diffraction and interference. In 1927, Davisson and Germer performed an experiment in which diffraction of electrons through a nickel crystal (Bragg scattering) was observed. This kind of experiment has been repeated with protons, neutrons, helium atoms, ions, the wave relation for material particles always being verified. Let us notice that the De Broglie relation implies that interference/diffraction effects could be observed not only when considering a beam of particles, but also for single ones. The idea of devising an experiment to look at the interference pattern created by the passage of a single electron through two slits (such as in the classical Young experiment for light) dates back to a proposal of Schroedinger. Feynmann uses it to introduce the reader to the fundamental concepts of QM, identifying it as "a phenomenon which is impossible […] to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the only mystery [of quantum mechanics]." [15], but warns the readers not to believe that such an experiment could ever be performed. On the contrary, following
a series of newly developed electron microscopy techniques and some clever innovation, the experiment was done first in 1972 by Merli et al. [16] in Bologna and in 1976 by Tonomura et al. [20] in Tokyo*. Let us examine the experiment in more detail. The source of electrons is set up in such a way that only one electron hits the detector (screen), at each time. Every electron hits the screen in a specific position, leaving a point-like image, but different electrons hit the screen at different points, with a pattern that initially looks completely random. Only after several thousands hits, one can start to recognise a band pattern, which looks the same as that of interference for light. Fig. 1 shows the creation of such a pattern as more and more electrons are shot (taken from [16]). Every time one repeats the experiment, the pattern is reconstructed out of a different sequence of points. In these results, it is easy to recognize a manifestation of the particle-wave duality principle: an interference pattern, typical of wave-like objects, forms on the screen even if every electron leaves a spot on the screen as if it were a point-like particle. But it is also possible to infer another intrinsic characteristic of QM: its probabilistic nature. Indeed, the position at which each electron hits the screen cannot be determined in a deterministic way, but there are areas of the screen where there is a higher/lower probability for the electron to get (bright/dark fringes).

To conclude this brief overview of the interference experiment in QM, it is important to underline that if we try to determine through which slit the electron has gone through

* The story of this experiment, which was defined as the most beautiful one in physics by the journal "Physics World" after questioning its readers, can be found in the website: http://l-esperimento-piu-bello-della-fisica.bo.imm.cnr.it/.
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(see [19] for description of a possible experimental realization), the interference pattern disappears, the electrons hitting the screen only right behind the two slits, as classical bullets would do.

I would like now to introduce one more kind of experiments which, together with what said up to now, can provide a feeling about some phenomenology that has lead physicists to formulate the set of postulates that govern QM, as we will state in the next section. Such experiments can actually be done either with photons and polarization filters (which are such to discriminate between the two possible polarizations of the photon) or with spin-1/2 atoms and Stern-Gerlach apparatus (which is able to discriminate between the two possible values of the intrinsic angular momentum). Fig. 2(a) shows the scheme of the latter set-up, with the magnetic field of the apparatus pointing along the z-direction: the atoms passing through it are separated into two beams. Spin represents the internal degree of freedom of our particles, the only one we are looking at in these experiments, neglecting all other dynamical variables, such as position. Contrary to classical-like variables, spin can take in our examples only two possible values (that we will denote with + and −). We cannot predict in a determinist way where each atom goes, i.e. with value of the spin it has, but (by repeating the

![Figure 2](image)

**Figure 2.** a) Scheme of a Stern-Gerlach experiment, with the magnetic field pointing in the z-direction. b) Scheme of an experiment in which two Stern-Gerlach apparatuses, with the magnetic field pointing in the z- and in the x-direction, are put in succession.
experiment many times) we can say that each atom has 50% of probability to be in the upper ($|+\rangle$) or lower ($|-\rangle$) beam*. Fig. 2(b) shows what happens if we enrich the experiment by letting the atoms go first through an apparatus with the magnetic field along the z-direction and then letting one of the beams (say the upper one) go through a second apparatus with the magnetic field along the x-direction: the upper beam is further split in two, again with a 50% of probability. Now, we can perform a final experiment where we put in succession three apparatuses, with the magnetic field pointing along Z, X and Z again, and where we recombine the two beams coming out of the second (X) apparatus before let them go through the last one, as shown in Fig. 3.

![Scheme of the experiment with three Stern-Gerlach apparatuses, explained in the text.](image)

On the basis of the previous experiments and if we follow classical probability rules, we expect to see finally half of the atoms in the upper and half of them in the lower beam (see Fig. 4(a) for a sketch of the calculation). On the contrary, if the experiment is performed, the atoms end up to be in the upper (lower) beam with a 100% (0%) probability. Obviously, these results cannot be understood classically, even if we accept that a quantum object has an intrinsically aleatory behaviour: we have also to modify the probability laws we use.

To get out of this apparent paradox, we have to give up our classical intuition, look at the experiments carefully and build a new conceptual framework out of them. Without going into all details, the key ingredients for explaining the described quantum phenomena might be summarised in the two following concepts:

**Superposition of states.** At no step in our experiment, we can assume that our quantum object is in one of the two possible alternatives representing their state (left/right polarization, +/- spin, first/second slit) but we can specify with what probability the object might be in one of the two possible alternatives; thus we may say that the object is in a superposition of the possible (classical) alternatives, with the coefficients of the superposition being related to the corresponding probability (this assumption incorporates the fact that interference emerge in linear phenomena, for which the superposition principle applies); a detailed analysis of

* The careful reader may have noticed that, in the interference experiment, electrons could choose between two possible alternatives (slits). We suggest to look back at this analogy after having read the rest of this section.
all mentioned experiments shows that the coefficients are to be chosen in the set of complex numbers.

**Probability amplitude.** Probability laws (that require multiplication for successive independent events and a sum over the set of possible independent outcomes) have to be applied to the coefficients of the superposition, but the latter cannot be identified with probabilities. Indeed, to get the probability of a given alternative, one has to take the modulus of the, generally complex, coefficient, which is therefore identified with a probability amplitude.

Let us observe that one needs only some basic knowledge of algebra to recognise that the possible classical alternatives (in our examples two, but the discussion can be easily generalised to other cases) may be thought of as a set of independent and orthogonal vectors, i.e. a basis for the linear space of all possible states, with complex coefficients, the probability of finding one of them being given by the square of orthogonal projection of the state along the considered element of the basis. Here, for lack of time and simplicity, we have chosen to consider in detail the problem of defining what we mean by the set of states in QM. Similar arguments may be used to study the problem of defining what we mean by observables, as well as by the process of measure. From all these considerations, it emerges that the natural mathematical framework—which will be given in full generality in the next section—is that of (complex, separable) Hilbert spaces.

As a final exercise for this section, we invite the reader to use this concepts to show the results of the last experiment with the Stern-Gerlach apparatus, as sketched in Fig. 4(b), and therefore describe it as an interference phenomenon.
1.2. Fundamentals of the new theory

We are ready now to formulate what are called the "Postulates" of QM. They give the mathematical framework and the conceptual physical interpretation to set up a QM problem. We will try to illustrate them by means of some simple but paradigmatic examples. As said in the introduction, we will skip the problem of defining quantum measurements, which, despite being very interesting and intriguing, will lead us too far away from the main topics of these lectures. We refer the interested reader to QM textbooks [5, 10] or books on Quantum Information and Computation [14].

1.2.1. The space of states

A physical (pure) state of a QM system is represented by rays in a complex and separable Hilbert space $\mathcal{H}$, i.e. equivalence classes of unit vectors under the relation $|\psi_1\rangle \equiv |\psi_2\rangle \iff |\psi_2\rangle = e^{i\alpha} |\psi_1\rangle$, $\alpha \in \mathbb{R}$.

We remark that linearity and complex field are needed in order to accommodate all interference phenomena described in the previous section, while a scalar product is necessary to define the notion of probability. The vectors are to be normalised to one since their norm gives the total probability. Also, multiplying by an overall pure phase factor does not change the physical content. Separability is necessary to have a finite or denumerable complete set of orthonormal vectors, according to which we can decompose the state in a linear superposition of possible alternatives.

In the following we will use Dirac notation and represent a vector $\psi \in \mathcal{H}$ with the "ket" $|\psi\rangle$, while its dual $\psi^* \in \mathcal{H}^*$ is represented through the "bra" $\langle \psi |$, in such a way that the scalar product $(\psi, \phi)$ between any two vector can be represented (according to Riesz theorem) as the "bracket": $\langle \psi | \phi \rangle$.

A ray in $\mathcal{H}$ can be univocally determined by the projection operator:

$$\rho_{\psi} \equiv |\psi\rangle \langle \psi|$$

(7)

where we have supposed (as we will do in the following, if not specified differently) that the vector is normalized. We recall that $\rho_{\psi}$ is a bounded, positive semi-definite, trace-one operator such that $\rho_{\psi}^2 = \rho_{\psi}$. This means also that it is a rank-one operator.

As we will see in some examples below, in many physical problems it is interesting to consider the possibility that a system might be prepared not in a unique state, but in a statistical mixture or mixed state, i.e. a collection of states $\{|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_N\rangle\}$ with, respectively, probabilities $p_1, p_2, \cdots, p_N$. Such a state is represented by a so-called density matrix operator, defines as:

$$\rho \equiv \sum_{j=1}^{N} |\psi_j\rangle \langle \psi_j|$$

(8)

Again, $\rho$ is bounded, positive semi-definite and trace-one, but now $\rho^2 \neq \rho$ (for $N > 1$). Notice that it has rank $N$. 
Examples. The simplest example one can consider is that of a two-level system, able to describe, e.g., the polarization degrees of freedom of a photon or the spin of an electron. In this case $H = \mathbb{C}^2$.

More generally one can consider an $N$-level system, whose space of states is given by $\mathbb{C}^N$.

In the case of infinite dimensions, $H$ is often realized as the space of square-integrable functions over an open domain $D \in \mathbb{R}^n$, the latter representing the classical configuration space of the system:

$$H = \{ \psi(x) : \int_D dx |\psi(x)|^2 < \infty \} \equiv L^2(D)$$

In this case, $\rho(x) \equiv |\psi(x)|^2$ might be interpreted as a probability density, $|\psi(x)|^2 dx$ representing the probability of finding our system in the volume $[x, x + dx]$ of the classical configuration space. The function $\psi(x)$ is called wave-function.

1.2.2. Observables

The space of observables $O$ of a QM system given by the set of all self-adjoint operators on $H$.

Thus, every observable $\hat{O}$ has a real spectrum and admits a spectral decomposition:

$$\hat{O} = \sum_{\lambda} \lambda \hat{P}_\lambda, \quad \hat{P}_\lambda = \equiv |\psi_\lambda\rangle \langle \psi_\lambda|$$

where $\{|\psi_\lambda\rangle\}$ are the complete set of eigenvectors of $O$, with eigenvalue $\lambda^*$. Since they form an orthonormal basis for $H$, we have the relations,

$$\hat{P}_\lambda \hat{P}_\mu = \delta_{\lambda\mu} \hat{P}_\lambda$$

$$\sum_{\lambda} \hat{P}_\lambda = \mathbb{I}$$

Given a pure state $|\psi\rangle \in H$, one defines the mean value of an observable $O$ as:

$$\langle \hat{O} \rangle \equiv \langle \psi | \hat{O} | \psi \rangle = \text{Tr} \left[ \rho_\psi \hat{O} \right]$$

Notice that (13) defines a paring between $H$ and $O$ (quadratic in the vectors and linear in the observables), which depends on the scalar product.

Also, we may consider the variance of $\hat{O}$ over the state $|\psi\rangle$ as given by:

$$\Delta \hat{O} \equiv \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$$

* Strictly speaking, in writing this expression we have assumed the spectrum to be discrete. But an analogous formula holds in the case of continuous spectra with the sum replaced by the integral over the spectral measure. For simplicity of notation, in the rest of the paper we write only use the sums, assuming they have to be replaced by integrals when the spectrum is continuous and the true eigenvectors $|\psi_\lambda\rangle$ have to be replaced by generalized ones.
More generally, one can define the mean value and variance of $O$ over a mixed state by taking the average over the statistical mixture, i.e.:

$$\langle \hat{O} \rangle \equiv \sum_{j=1}^{N} p_j \text{Tr} \left[ \rho_{\psi_j} \hat{O} \right] = \text{Tr} \left[ \rho \hat{O} \right]$$ (15)

**Example: Position and Momentum.** Let us consider $\mathcal{H} = L^2(\mathbb{R}) = \{ \psi(x) \}$. On this space we can define two important operators, the position $\hat{x}$ and the momentum $\hat{p}$ given by respectively the multiplication by the coordinate and the derivative with respect to it:

$$\hat{x} \psi(x) = x \psi(x)$$
$$\hat{p} \psi(x) = -i \hbar \frac{d}{dx} \psi(x)$$ (16)

which satisfy the so called canonical commutation relations (CCR):

$$[\hat{x}, \hat{p}] = i \hbar \mathbb{I}$$ (17)

which look the same as the classical ones, if we replace the Poisson brackets with the Lie commutator*. It is said that eq.ns (16) realize the CCR in the coordinate representation. Notice that relation (17) implies that neither $\hat{x}$ nor $\hat{p}$ can be realised as bounded operators on some Hilbert space: thus one need to specify their domains $\mathcal{D}_x, \mathcal{D}_p \subset \mathcal{H}$ on which they are self-adjoint. Usually, one chooses $\mathcal{D}_x = \mathcal{D}_p = \mathcal{S}(\mathbb{R})$, the space of Schwartz functions. The spectrum of both these operators is continuous and equal to $\mathbb{R}$, while the corresponding generalized eigenfunctions are respectively given by:

$$|x_0\rangle = \delta(x - x_0) \text{ with eigenvalue } x_0 \in \mathbb{R}$$ (18)
$$|p_0\rangle = e^{ip_0x} \text{ with eigenvalue } p_0 \in \mathbb{R}$$ (19)

Let us recall that the Fourier transform:

$$\psi(p) \equiv \frac{1}{\sqrt{2\pi}} \int dx e^{ipx} \psi(x)$$ (20)

is a unitary operator on $L^2(\mathbb{R})$. Thus, we may work as well on $\mathcal{H} = \{ \psi(p) \} (\sim H)$, where the $\hat{x}$ and the $\hat{p}$ operators are now given by:

$$\hat{x} \psi(p) = -i \hbar \frac{dp}{d} \psi(p)$$
$$\hat{p} \psi(p) = p \psi(p)$$ (21)

These define the so-called momentum representation of the CCR.

Finally, we observe that this construction can be easily generalized to higher dimensions, i.e. to $\mathcal{H} = L^2(\mathbb{R}^n)$, by defining two sets of operators $\hat{x}_j$ and $\hat{p}_j$ ($j = 1, 2, \cdots, N$), such that $[\hat{x}_j, \hat{p}_k] = i \hbar \delta_{jk} \mathbb{I}$. □

* We will come back to the problem of recovering the classical structures starting from the quantum one later on in these lectures.
Example: two-level/fermionic system. In the finite-dimensional case all self-adjoint operators are bounded and observables are represented by $N \times N$ complex matrices: $\mathcal{O} = \mathbb{M}_{N}$. For a two-level system, a basis of all hermitian operators is given by the identity $\mathbb{I}$ and the set of the three Pauli matrices $\sigma_{\alpha}$, ($\alpha = 1, 2, 3$):

$$
\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

(22)

It is not difficult to see that the operators $S_{\alpha} = \hbar \sigma_{\alpha}$ yields the fundamental representation (i.e. spin 1/2) of the $SU(2)$ algebra:

$$
[S_{\alpha}, S_{\beta}] = i\hbar \epsilon^{\alpha\beta\gamma} S_{\gamma}
$$

(23)

The canonical vectors:

$$
|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

(24)

are the eigenstates of the $S_{3}$ operator, with eigenvalues $\pm \hbar/2$. It is convenient to define the two ladder operators:

$$
\sigma^{\pm} = \sigma_{1} \pm i\sigma_{2}
$$

(25)

satisfying the algebra commutators:

$$
[\sigma^{+}, \sigma^{-}] = \sigma^{z}, \quad [\sigma^{z}, \sigma^{\pm}] = 2\sigma^{\pm}
$$

(26)

and the anti-commutation relations:

$$
(\sigma^{+})^{2} = (\sigma^{-})^{2} = 0, \quad \{\sigma^{+}, \sigma^{-}\} = \mathbb{I}
$$

(27)

From the latter, it follows immediately that $H$ is generated by the two states: $|0\rangle \equiv |\rangle$ and $|1\rangle \equiv \sigma^{+}|0\rangle = |+\rangle$, while $\sigma^{+}|1\rangle = 0$ as well as $\sigma^{-}|0\rangle = 0$. The operators $\sigma^{\pm}$ satisfying (27) are called fermionic creation/annihilation operators, while $|0\rangle$ is interpreted as the vacuum state. We can also define the number operator: $N \equiv \sigma^{+}\sigma^{-}$, for which $|0\rangle, |1\rangle$ are eigenvectors with eigenvalues 0, 1 respectively. This represents an algebraic way of encoding Pauli exclusion principle, which states that two fermions cannot occupy the same state.

A two-level system is what is a called a qubit in the context of quantum information theory [14]. □

Example: bosonic systems. Bosonic creation/annihilation operators are defined as two operators $a, a^{\dagger}$ on $\mathcal{H}$ such that:

$$
[a, a^{\dagger}] = \mathbb{I}
$$

(28)

Notice that this relation implies that the operator $a$ and its adjoint $a^{\dagger}$ cannot be bounded. Hence $\mathcal{H}$ has to be infinite dimensional. It is useful to define the number operator $N \equiv a^{\dagger}a$, which is self-adjoint and satisfies: $[N, a] = a$, $[N, a^{\dagger}] = a^{\dagger}$. It is not difficult to show [5, 10] that $N$ is such that:

- its spectrum is composed of all integers: $\sigma(N) = \{0, 1, 2, \cdots\}$;
- its eigenvector $|0\rangle$ corresponding to the lowest eigenvalue $\lambda_0 = 0$ is such that: $a |0\rangle = 0$;
- its eigenvector $|n\rangle$ corresponding to the $n-th$ eigenvalue $\lambda_n = n$ is given by:

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |\psi_0\rangle$$

(29)

Also, the following relations hold:

$$a^\dagger |\psi_n\rangle = \sqrt{n+1} |\psi_{n+1}\rangle$$

(30)

$$a |\psi_n\rangle = \sqrt{n} |\psi_{n-1}\rangle$$

(31)

showing that $a$ ($a^\dagger$) moves from one eigenstate to the previous (next) one, acting as ladder operators. The complete set of normalised eigenstates $\{|n\rangle\}$ provide a basis for $\mathcal{H}$, which is called Fock basis.

It might be interesting to look for the coordinate representation of (28). On $\mathcal{H} = L^2(\mathbb{R}) = \{\psi(x)\}$, one has:

$$a = \frac{1}{\sqrt{2}} \left( x + \frac{d}{dx} \right)$$

(32)

$$a^\dagger = \frac{1}{\sqrt{2}} \left( x - \frac{d}{dx} \right)$$

(33)

while:

$$|0\rangle = C_0 e^{-x^2/2}$$

(34)

$$|n\rangle = C_n e^{-x^2/2} P_n(x)$$

(35)

where $P_n(x)$ is the $n$-th Hermite polynomial, and $C_n$ normalization constants. As a final remark, which will be useful in the following, we notice that, using (16), it is easy to show that:

$$a = \frac{1}{\sqrt{2}} (\hat{x} + i\hat{p})$$

$$a^\dagger = \frac{1}{\sqrt{2}} (\hat{x} - i\hat{p})$$

(36)

As we will see in the next subsection, these operators are the one necessary to describe the quantum 1D harmonic oscillator.

**Example: composite systems.** For completeness, we very briefly review what happens when the system we consider is composed of $N$ independent degrees of freedom/particles*. In this case, the Hilbert space of the total system is given by the tensor product of the Hilbert space of each particle: $\mathcal{H} = \otimes_{j=1}^N \mathcal{H}_j$. States in $\mathcal{H}$ of the form:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_N\rangle \quad \text{with} \quad |\psi_j\rangle \in \mathcal{H}_j$$

(37)

are said to be separable. A state that cannot be written as so, is called entangled.

Entanglement is a truly quantum property: it encodes the possibility of knowing some properties of one subsystem by measuring observables on the other part. Consider for

* We are assuming that the particles are distinguishable.
example two spin-1/2 particles, that are usually called $A$ and $B$, in the so called singlet state (we omit the symbol of tensor product):

$$|\psi\rangle = \frac{|+\rangle_A |-\rangle_B + |-\rangle_A |+\rangle_B}{\sqrt{2}} \quad (38)$$

In this state, the spins of neither of $A$ nor of $B$ are defined, but $A$ and $B$ have opposite spins. When an experimentalist, say Alice, performs a measurement on the system $A$ of the spin along the third component (she measures the operator $S_3$), she finds $+\hbar/2$ or $-\hbar/2$ with 50% probability, but a second physicist, say Bob, can infer without performing any measurement on it that the quit $B$ d has the opposite value of the spin. This phenomenon, that Einstein himself defined it as "that spooky action at a distance", is at the origin of much work and interesting discussions in the history of development of QM (the EPR paradox, the theory of hidden variables and Bell’s inequality, for example) and it is now at the heart of recent applications in the field of quantum information and quantum computation, such as teleportation. The interesting reader can look at [14].

As a final remark, let us observe that different states might encode a different level of entanglement; for examples the singlet state given above is maximally entangled. Indeed we can measure the entanglement of a state by using physical quantities, called entanglement witnesses. For examples, for a pure state described by a density matrix $\rho_{AB}$, one can use the von Neumann entropy:

$$S(\rho_{AB}) \equiv \text{Tr}_A[\rho_A \log \rho_A] \quad (39)$$

where $\rho_A = \text{Tr}_B[\rho_{AB}]$ is the partial trace over the system $B$ (of course one can interchange the role of $A$ and $B$ to get the same result). We will not further discuss this topic and refer the interested reader for example to [4], where it is also possible to find a discussion of the geometric aspects of entanglement.

1.2.3. Dynamical evolution

The dynamics of a quantum state $|\psi(t)\rangle \in \mathcal{H}$ is specified by a suitable self-adjoint operator $\hat{H}$, called Hamiltonian, and governed by the Schroedinger equation:

$$\frac{i\hbar}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \quad (40)$$

In the following we will always assume that $\hat{H}$ is time-independent. In this case, $\hat{H}$ being self-adjoint, the equation can be solved by introducing the (strongly-continuous) one-parameter group of unitary operators (the evolution operator):

$$U(t) = e^{-\frac{i\hat{H}t}{\hbar}} \quad (41)$$

since then:

$$|\psi(t)\rangle = U(t) |\psi(0)\rangle \quad (42)$$
Notice that, since $U(t)$ is unitary, scalar products hence probabilities are conserved. Indeed, some algebra shows that we can derive a sort of continuity equation of the form:

$$\frac{d}{dt}⟨ψ|ψ⟩ = -\frac{i\hbar}{\hbar} \left( ⟨ψ\hat{H}|ψ⟩ - ⟨ψ|\hat{H}ψ⟩ \right)$$

(43)

It is also easy to prove that the (eventually generalized) eigenvectors of $\hat{H}$ are (the only) stationary states:

$$\hat{H} |ψ_\lambda⟩ = \lambda |ψ_\lambda⟩ \Leftrightarrow |ψ_\lambda(t)⟩ = e^{-\frac{iE_\lambda t}{\hbar}} |ψ_\lambda⟩$$

(44)

Using the fact the eigenstates form an orthonormal basis in $\mathcal{H}$, it is finally easy to see that the evolution of any state can be written in the form:

$$|ψ(t)⟩ = \sum_\lambda c_\lambda e^{-\frac{iE_\lambda t}{\hbar}} |ψ_\lambda⟩$$

(45)

**Example: Free particle on $\mathbb{R}^n$.** This system is easily described in the coordinate representation and is given by the Hamiltonian:

$$\hat{H}_0 = \frac{\hat{\vec{p}}^2}{2m} = -\frac{\hbar}{2m} \nabla^2$$

(46)

where $m$ is the mass of the particle. Notice that, as function of the momentum, the Hamiltonian has the same expression as in the classical case: to obtain it we just need to replace the function $\vec{p}$ with the operator $\hat{\vec{p}}$. This can be considered as first rule of quantization.

Its (continuous) eigenvalues and (generalized) eigenfunctions are easily found to be:

$$E_p = \frac{\hbar|\vec{p}|^2}{2m}, \quad ψ_p(\vec{x}) = e^{i\vec{p}·\vec{x}}$$

(47)

In this case, the continuity eq. (43) assumes a particularly nice and transparent expression:

$$\frac{∂J_0}{∂t} + \vec{∇} \cdot \vec{J} = 0$$

(48)

where the probability density $J_0$ and the probability current $\vec{J}$ are respectively given by:

$$J_0(x,t) = |ψ(x,t)|^2$$

$$\vec{J}(x,t) = \frac{\hbar}{m} \left[ ψ^* \vec{∇} ψ - ψ \vec{∇} ψ^* \right]$$

(49)

(50)

**Example: The 1D harmonic oscillator.** Many interesting dynamical systems are described by hamiltonians of the form:

$$H = \frac{\vec{p}^2}{2m} + V(\vec{x})$$

(51)

representing a particle in an external potential $V$. In the previous example, we have seen that, to get the quantum version of a free particle, we need just to replace the
momentum $\vec{p}$ with the operator $\hat{\vec{p}}$ in the classical Hamiltonian function. In a similar way, one is led to consider the quantum version of (51) to be given by:

$$H = \frac{\hat{\vec{p}}^2}{2m} + V(\hat{\vec{x}})$$  \hfill (52)

To see to what extent we can use thin approach, let us consider the 1D harmonic oscillator whose classical Hamiltonian is given by (we work in suitable units so that $\{x, p\} = 1/\hbar$):

$$H = \frac{\hbar \omega}{2} (p^2 + x^2)$$  \hfill (53)

Thus we would like to consider the quantum Hamiltonian:

$$\hat{H} = \frac{\hbar \omega}{2} (\hat{p}^2 + \hat{x}^2)$$  \hfill (54)

One could notice that (53) could be rewritten in different equivalent ways:

$$H = \frac{\hbar \omega}{2} (p^2 + x^2) = \frac{\hbar \omega}{2} (x + ip)(x - ip) = \frac{\hbar \omega}{2} (x - ip)(x + ip)$$  \hfill (55)

since $x, p$ are commuting functions. At the quantum level, however, this does not longer hold true (see (17)) and one would obtain different quantum Hamiltonian (specifically, they differ by constants). In textbooks, this ambiguity in the process of quantization is resolved by introducing the so-called ”symmetrization postulates”: to obtain the quantum Hamiltonian $\hat{H}$ out of the classical one $H$, any monomial in the variables $x, p$ that classically reads as $x^m p^n$ has to be replaced with the symmetric version obtained by writing the products of $n$ times $x$-factors and $m$ times $p$-factors in all possible orders. Applying this rule to our problem, we exactly obtain eq. (54), that can be also written in the coordinate representation as:

$$\hat{H} = \frac{\hbar \omega}{2} \left( -\frac{d^2}{dx^2} + x^2 + 1 \right)$$  \hfill (56)

Also, we can make use of (36), to rewrite it as:

$$\hat{H} = \frac{\hbar \omega}{2} \left( a^\dagger a + \frac{1}{2} \right) = \frac{\hbar \omega}{2} \left( N + \frac{1}{2} \right)$$  \hfill (57)

We see that the Hamiltonian of the 1D harmonic oscillator is (up to a constant) the number operator of bosonic type, whose spectral problem has been solved in the previous subsection. □

1.2.4. QM Pictures and the GNS construction

The structure of QM that we have outlined above is fitted to describe the so-called Schroedinger picture in which the relevant object out of which one construct the whole theory is the Hilbert space $\mathcal{H}$. Such a picture is also called ”wave QM” since, for a particle say in $\mathbb{R}^n$, the starting point is a wave function $\psi(x, t) \in L^2(\mathbb{R})$ whose
evolution is governed by the differential equation given by the Schroedinger one. For example, if the Hamiltonian is of type (52), one has:

$$i\hbar \frac{d}{dt}\psi(x,t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{x}) \right] \psi(x,t)$$

(58)

The most general solution is given by (see eq. (45)):

$$\psi(x,t) = \sum_{\lambda} c_{\lambda} e^{-\frac{iE_{\lambda}}{\hbar} t} \psi_{\lambda}(x,t)$$

(59)

where \( \psi_{\lambda}(x,t) \) satisfy the eigenvalue equation:

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{x}) \right] \psi_{\lambda}(x) = E_{\lambda} \psi_{\lambda}(x)$$

(60)

However, historically the ”wave picture” developed by Schroedinger was paralleled by the so called Heisenberg picture, in which the key ingredient becomes the space of observables \( \mathcal{O} \), from which one can recover (thanks to the ”duality” that we will describe shortly below) the space of states \( \mathcal{H} \). In this picture, a state \( \ket{\psi} \) become time-indipendent while an operator \( \hat{O} \) evolves according to the operator version of the Schroedinger equation:

$$\frac{d}{dt} \hat{O}(t) = \frac{i}{\hbar} \left[ \hat{H}, \hat{O}(t) \right]$$

(61)

i.e. to

$$\hat{A}(t) = U(t)^\dagger \hat{A} U(t)$$

(62)

This picture is also called ”matrix QM”, since Heisenberg got it for the first time by introducing infinite matrices that where representing the possible transitions between energy levels in atoms.

The possibility of recovering the Hilbert space out of the algebra of operators stands on the so-called Gelfand-Naimark-Segal or GNS-construction [3], which is also at the heart of the algebraic approach to quantum field theory [11].

The starting point of this construction is an abstract \( \mathbb{C}^* \)-algebra \( \mathcal{A} \) with unity \( \mathbb{1} \). The elements \( a \in \mathcal{A} \) such that: \( a = a^\dagger \) constitute the real vector space set \( \mathcal{A}_{re} \), which represents the algebra of observables. Also, \( \mathcal{A}_{im} = i\mathcal{A}_{re} \). Clearly: \( \mathcal{A} = \mathcal{A}_{re} \oplus \mathcal{A}_{im} \) as it can be seen from the obvious decomposition: \( a = a_1 + ia_2 \), with:

$$a_1 = \frac{a + a^\dagger}{2}, \quad a_2 = \frac{a - a^\dagger}{2i}$$

(63)

We define now the space of states, \( \mathcal{D}(\mathcal{A}) \), over the \( \mathbb{C}^* \)-algebra \( \mathcal{A} \) as the space of linear functionals \( \omega : \mathcal{A} \to \mathbb{C} \) that are:

- real: \( \omega (a^\dagger) = \overline{\omega (a)} \ \forall a \in \mathcal{A} \),
- positive: \( \omega (a^\dagger a) \geq 0 \ \forall a \in \mathcal{A} \) and
- normalized: \( \omega (\mathbb{1}) = 1 \)
Each functional $\omega$ defines a non-negative pairing $\langle \cdot | \cdot \rangle_\omega$ between any two elements $a, b \in \mathcal{A}$ via:

$$\langle a | b \rangle_\omega \equiv \omega(a^\dagger b) \quad (64)$$

Reality and positivity of the state guarantee that the pairing (64) satisfies the Schwartz inequality, $|\langle a | b \rangle_\omega| \leq \sqrt{\langle a | a \rangle_\omega \langle b | b \rangle_\omega}$, but the pairing might be degenerate. We are thus led to consider the Gelfand ideal $\mathcal{I}_\omega$ consisting of all elements $j \in \mathcal{A}$ such that $\omega(j^\dagger j) = 0$ and to define the set $\mathcal{A}/\mathcal{I}_\omega$ of equivalence classes:

$$\Psi_a =: [a + \mathcal{I}_\omega] \quad (65)$$

It is immediate to see that $\mathcal{A}/\mathcal{I}_\omega$ is a pre-Hilbert space with respect to the scalar product:

$$\langle \Psi_a, \Psi_b \rangle = \omega(a^\dagger b) \quad (66)$$

Completing this space with respect to the topology defined by the scalar product, one obtains a Hilbert space $\mathcal{H}_\omega$ on which the original $\mathbb{C}^\ast$-algebra $\mathcal{A}$ acts via the following representation:

$$\pi_\omega(a)\Psi = \Psi_{ab} \quad (67)$$

Notice that the equivalence class $\Omega = \Psi_1$ of the unit element satisfies: $||\Psi_1|| = \sqrt{\langle \Psi_1 | \Psi_1 \rangle} = 1$ and provides a cyclic vector* for the representation $\pi_\omega$. Moreover: $\langle \Omega | \pi_\omega(a) | \Omega \rangle = \omega(a)$.

This tells us that, if we consider that $\mathcal{A}$ acts by duality on $\mathcal{D}(\mathcal{A})$, the Hilbert space corresponding to a given state $\omega$ is the orbit of $\mathcal{A}$ through $\omega$ itself. Notice that any other element $b \in \mathcal{A}$ such that the vector $\Psi = \pi_\omega(b)\Omega$ is of unit norm, defines a new state $\omega_\Psi$ by:

$$\omega_\Psi(a) = \langle \Psi | \pi_\omega(a) | \Psi \rangle = \omega(b^\dagger ab) \quad (68)$$

These states are called vector states of the representation $\pi_\omega$, and are particular examples of more general states of the form: $\omega_\rho(a) = Tr[\rho \pi_\omega(a)]$, where $\rho \in \mathcal{B}(\mathcal{H}_\omega)$ is a density operator. One says that a state $\omega$ is pure iff it cannot be written as a convex combination of other states in $\mathcal{D}(\mathcal{A})$, so that the set of pure states $\mathcal{D}^1(\mathcal{A})$ defines a set of extremal points in $\mathcal{D}(\mathcal{A})$.

The universality and uniqueness of the GNS construction is guaranteed by the following theorems [3]:

(i) If $\pi_\omega$ is a cyclic representation of $\mathcal{A}$ on $\mathcal{H}$, any vector representation $\omega_\Psi$, for a normalized $\Psi$, is equivalent to $\pi_\omega$.

(ii) A GNS representation $\pi_\omega$ of $\mathcal{A}$ is irreducible iff $\omega$ is a pure state.

**Example.** A typical example of a $\mathbb{C}^\ast$-algebra is the algebra $\mathcal{B}(\mathcal{H})$ of the bounded operators on a Hilbert space $\mathcal{H}$. In this case $\mathcal{A}_{re} \equiv \mathcal{B}_{sa}(\mathcal{H})$, the set of the bounded self-adjoint operators on $\mathcal{H}$.

* We recall that a vector $\Omega \in \mathcal{H}_\omega$ is called cyclic if $\pi_\omega(\mathcal{A})$ is dense in $\mathcal{H}_\omega$. 

The GNS construction is very simple in the finite-dimensional case, i.e. for the algebra \( \mathcal{A} = \mathbb{M}_n \). Any non-negative operator \( \omega \in \mathcal{A} \) defines a state by:

\[
\omega(A) = \text{Tr}[\omega A], \quad \forall A \in \mathcal{A}
\]

while we can define the scalar product in \( H_\omega \) as:

\[
\langle A|B \rangle = \omega(A^*B) = \text{Tr}[B\omega A^*]
\]

If \( \omega \) is a rank-1 projector and \( \{e_k\} \) is an orthonormal basis for which \( \omega = |e_1\rangle\langle e_1| \), writing \( A_{km} \) for the matrix elements of \( A \) in such a basis, the scalar product assumes the form:

\[
\langle A|B \rangle = \sum_{k=1}^n \bar{A}_{k1} B_{k1}
\]

while the Gelfand ideal \( \mathcal{I}_\omega \) is given by: \( \mathcal{I}_\omega = \{X \in \mathcal{A} : X_{k1} = 0, \ k = 1, \ldots, n\} \). Thus \( \mathcal{H}_\omega = \mathcal{A}/\mathcal{I}_\omega \) is nothing but \( \mathbb{C}^n \) itself with \( \pi_\omega \) being the defining representation.

More generally, if \( \omega \) is a rank-\( m \) density operator: \( \omega = p_1|e_1\rangle\langle e_1| + \cdots + p_m|e_m\rangle\langle e_m| \), with \( p_1, \cdots p_m > 0; \ p_1 + \cdots p_m = 1 \), the scalar product is given by:

\[
\langle A|B \rangle = \sum_{k=1}^n \sum_{j=1}^m p_m \bar{A}_{kj} B_{kj}
\]

The Gelfand ideal is given by: \( \mathcal{I}_\omega = \{X \in \mathcal{A} : X_{kj} = 0, \ k = 1, \ldots, n; \ j = 1, \ldots, m\} \), showing that \( \mathcal{H}_\omega = \bigoplus_m \mathbb{C}^n \). Now the representation \( \pi_\omega \) is no longer irreducible, decomposing into the direct sum of \( m \) copies of the defining representation: \( \pi_\omega(A) = \mathbb{1}_m \otimes \mathcal{A} \).

2. Geometric Quantum Mechanics

To describe the geometry of QM, we will stick to the case of a finite \( n \)-level system, so that \( \mathcal{H} = \mathbb{C}^n \). We will use the material of ref. [9], where many more details (and references) may be found.

2.0.5. The Schroedinger equation as a classical dynamical system

We begin by considering the Schrödinger equation (40)\(^*\).

As \( \mathcal{H} \) is a vector space, there is a natural identification of the tangent space at any point \( \psi \in \mathcal{H} \) with \( \mathcal{H} \) itself: \( T_\psi \mathcal{H} \approx \mathcal{H} \), vectors in a Hilbert space playing a double rôle, as points of the space and as tangent vectors at a given point. More generally, we have the identification: \( T\mathcal{H} \approx \mathcal{H} \times \mathcal{H} \), with \( T\mathcal{H} \) the tangent bundle of \( \mathcal{H} \).

As in the case of differentiable manifolds, \( \psi = \psi(t) \) with \( \psi(0) = \psi \) will define a curve in \( \mathcal{H} \), and hence the quantity \( (d\psi(t)/dt)|_{t=0} \) will define the tangent vector at the

\(^*\) In the following we will write vectors as \( \psi, \phi, \cdots \), instead of using the Dirac notation \( |\psi\rangle, |\phi\rangle, \cdots \).
curve at $\psi \in \mathcal{H}$. A smooth assignment of tangent vectors at every point $\psi \in \mathcal{H}$ will define then a vector field, i.e. a smooth global section of $T\mathcal{H}$:

$$\Gamma : \mathcal{H} \to T\mathcal{H} ; \psi \mapsto (\psi, \phi), \psi \in \mathcal{H}, \phi \in T_\psi \mathcal{H} \approx \mathcal{H}$$  \hspace{2cm} (73)

We will employ the notation $\Gamma (\psi)$ for the vector field evaluated at the point $\psi$ with tangent vector at $\psi$ given by eq. (73).

Every vector field will define a derivation on the algebra of functions just as in the case of real manifolds. Specifically, if $\phi = (d\psi / dt) |_{t=0}, \psi (0) = \psi$ and $f : \mathcal{H} \to \mathbb{R}$ is a function, then:

$$\left( L_\Gamma (f) \right) (\psi) = \frac{d}{dt} f (\psi (t)) |_{t=0}$$  \hspace{2cm} (74)

will define the Lie derivative along $\Gamma$ on the algebra of functions. In local coordinates, i.e. choosing an orthonormal basis $\{e_i\}$, vectors (and tangent vectors) will be represented by $n$-tuples of complex numbers $\psi = (\psi^1, \cdots, \psi^n)$, $\psi^j \equiv \langle e_j | \psi \rangle$, and*:

$$\left( L_\Gamma (f) \right) (\psi) = \phi^i (\psi) \frac{\partial f}{\partial \psi^i} (\psi)$$  \hspace{2cm} (75)

Constant as well as linear vector fields will play a role in what follows. The former are characterized by $\phi = \text{const.}$ in the second argument of eq. (73), and give rise to the one-parameter group:

$$\mathbb{R} \ni t \mapsto \psi (t) = \psi + t\phi$$  \hspace{2cm} (76)

The latter are characterized instead by $\phi (\psi)$ being a linear and homogeneous function of $\psi$, i.e.: $\phi = A \psi$ for some linear operator $A$. In this case:

$$\psi (t) = \exp \{ tA \} \psi$$  \hspace{2cm} (77)

Of particular interest is the dilation vector field $\Delta$:

$$\Delta : \psi \mapsto (\psi, \psi)$$  \hspace{2cm} (78)

which corresponds to $A = \mathbb{I}$. In this case eq. (77) becomes:

$$\psi (t) = e^t \psi$$  \hspace{2cm} (79)

Eq. (78) exhibits clearly the fact that the dilation field leads to an identification of $\mathcal{H}$ with the fiber $T_\psi \mathcal{H}$. The latter carrying a natural linear structure, eq. (78) provides a tensorial characterization of the linear structure of the base space $\mathcal{H}$ by means of the vector field $\Delta$.

With every linear operator $A$ there is therefore associated the linear vector field

$$X_A : \mathcal{H} \to T\mathcal{H} ; \psi \mapsto (\psi, A \psi)$$  \hspace{2cm} (80)

which, in local coordinates, reads as:

$$X_A \equiv A^j_i \psi^j \frac{\partial}{\partial \psi^i}$$  \hspace{2cm} (81)

* As $\psi_j$ is complex: $\psi_j = q_j + ip_j$, $q_j, p_j \in \mathbb{R}$, the derivative here has to be understood simply as: $\partial / \partial \psi_j = \partial / \partial q_j - i \partial / \partial p_j$ (see also later on).
\( A^i_j \) being the representative matrix of the linear operator. In particular:

\[
\Delta = \psi^i \frac{\partial}{\partial \psi^i}
\]  
(82)

Notice however that, while linear operators form an associative algebra, vector fields do not: they form instead only a Lie algebra. An associative algebra can be recovered by using the same matrix \( A \) to define instead the \((1,1)\) tensor:

\[
T_A \equiv A^i_j d\psi^j \otimes \frac{\partial}{\partial \psi^i}
\]  
(83)

Then it is easy to check that the vector field \( X_A \) is recovered from \( T_A \) and the dilation field as:

\[
X_A = T_A (\Delta)
\]  
(84)

Coming back to the Schrödinger equation, the linear operator \( H \) will define a linear vector field that we will denote* for short as \( \Gamma_H \):

\[
\Gamma_H : \mathcal{H} \to T\mathcal{H} ; \Gamma_H : \psi \mapsto (\psi, -(i/\hbar) H\psi)
\]  
(85)

so that:

\[
\mathcal{L}_{\Gamma_H} \psi \equiv \frac{d}{dt} \psi = -\frac{i}{\hbar} H\psi
\]  
(86)

In this sense, the Schrödinger equation (40) can be viewed as a classical evolution equation on a complex vector space.

Let now \( h : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \) be a Hermitian structure on \( \mathcal{H} \), i.e.

\[
h (\phi, \psi) \equiv \langle \phi | \psi \rangle
\]  
(87)

define an Hermitian scalar product on \( \mathcal{H} \) with the usual properties.

If \( h \) is viewed more properly as a \((0,2)\) tensor field, then \( \phi \) and \( \psi \) in eq. (87) have to be viewed as tangent vectors at a point in \( \mathcal{H} \), and a more complete (albeit a bit more cumbersome) notation should be:

\[
h (\varphi) (\Gamma_\phi (\varphi), \Gamma_\psi (\varphi)) = \langle \phi | \psi \rangle
\]  
(88)

where \( h (\varphi) \) stands for \( h \) evaluated at point \( \varphi \in \mathcal{H} \). As the r.h.s. of this equation does not depend on \( \varphi \), this implies: \( \mathcal{L}_{\Gamma_H} \langle \phi | \psi \rangle = \mathcal{L}_{\Gamma_H} (h (\phi, \psi)) = 0 \) and, using eq. (40):

\[
0 = \mathcal{L}_{\Gamma_H} (h (\phi, \psi)) = (\mathcal{L}_{\Gamma_H} h) (\phi, \psi) + h (\mathcal{L}_{\Gamma_H} \phi, \psi) + h (\phi, \mathcal{L}_{\Gamma_H} \psi) =
\]

\[
= (\mathcal{L}_{\Gamma_H} h) (\phi, \psi) + \frac{i}{\hbar} \{ \langle H \phi | \psi \rangle - \langle \phi | H \psi \rangle \}
\]

which implies in turn, as \( H \) is self-adjoint, that:

\[
\mathcal{L}_{\Gamma_H} h = 0
\]  
(90)

i.e. that the Hermitian structure be invariant under the (unitary) flow of \( \Gamma_H \) (and viceversa), or, stated equivalently, that \( \Gamma_H \) be a Killing vector field for the Hermitian

* We use here the notation \( \Gamma_H \) instead of \( \mathbb{X}_H \) as a reminder of the fact that we had to include the "extra" factor \((-i/\hbar)\) in its definition.
structure. If instead the Hermitian structure is not invariant, then $H$ will fail to be self-adjoint w.r.t. the given Hermitian structure.

We can now decompose the Hermitian structure into real and imaginary parts as:

$$ h(\cdot, \cdot) = g(\cdot, \cdot) + i\omega(\cdot, \cdot) \tag{91} $$

where:

$$ g(\phi, \psi) = \frac{1}{2} \left[ \langle \phi | \psi \rangle + \langle \psi | \phi \rangle \right] \quad \text{and} \quad \omega(\phi, \psi) = \frac{1}{2i} \left[ \langle \phi | \psi \rangle - \langle \psi | \phi \rangle \right] \tag{92} $$

It is clear that both $g$ and $\omega$ are $(0, 2)$ tensors, and that $g$ is symmetric, while $\omega$ is skew-symmetric, hence a two-form. Eq. (90) implies then that both tensors are (separately) invariant under $\Gamma_H$. Notice that:

$$ \omega(\phi, i\psi) = g(\phi, \psi). $$

Hence, non-degeneracy of $h$ entails separately that of $\omega$ and of $g$. The non-degenerate two-form $\omega$ will be represented, in any one of the privileged charts, by a constant (and unitarily invariant) matrix. Hence it will be closed. Hence, $\omega$ will be a symplectic form, while $g$ will be a metric tensor.

Let now $\Gamma_H$ be a vector field of the form (85). Then, a little algebra shows that:

$$ (i\Gamma_H \omega)(\psi) = \omega \left( -\frac{i}{\hbar} H\phi, \psi \right) = \frac{1}{2\hbar} \left[ \langle H\phi | \psi \rangle + \langle \psi | H\phi \rangle \right] \tag{93} $$

On the other hand, if we define the quadratic function

$$ f_H(\phi) = \frac{1}{2\hbar} \langle \phi | H\phi \rangle \tag{94} $$

we can define its differential as the one-form:

$$ df_H(\phi) = \frac{1}{2} \left[ \langle \cdot | H\phi \rangle + \langle H\phi | \cdot \rangle \right] = \frac{1}{2} \left[ \langle \cdot | H\phi \rangle + \langle H\phi | \cdot \rangle \right] \tag{95} $$

the last passage following from $H$ being self-adjoint. Therefore:

$$ (i\Gamma_H \omega)(\psi) = df_H(\phi)(\psi) \forall \psi, \tag{96} $$

i.e. $\Gamma_H$ is Hamiltonian w.r.t. the symplectic structure, with the quadratic Hamiltonian $f_H$.

As a final remark, we recall that $\mathcal{H}$ is endowed with a natural complex structure $J$ defined simply by:

$$ J : \phi \rightarrow i\phi \tag{97} $$

with $J^2 = -\mathbb{I}$. Notice that:

$$ \omega(\phi, J\psi) = g(\phi, \psi) \tag{98} $$

The complex structure $J$ is then said to be compatible with the pair $(g, \omega)$ and the triple $(g, \omega, J)$ is called admissible. Thus, we can reconstruct the Hermitian structure as:

$$ h(\phi, \psi) = \omega(\phi, J\psi) + i\omega(\phi, \psi) = g(\phi, \psi) - ig(\phi, J\psi) \tag{99} $$

Notice also that:

$$ \omega(J\phi, J\psi) = \omega(\phi, \psi) \quad \text{as well as} \quad g(J\phi, J\psi) = g(\phi, \psi) \tag{100} $$

We can summarize what has been proved up to now by saying that $\mathcal{H}$ is a Kähler manifold, and that $h$ is the associated Hermitian metric, while $g$ is the Riemannian metric and $\omega$ the fundamental two-form. As $\omega$ is closed, $g$ is also a Kähler metric.
2.0.6. The projective Hilbert space

We have already seen that a physical state is not identified with a unique vector in some Hilbert space, but rather with a "ray", i.e. an equivalence class of vectors differing by multiplication through a nonzero complex number: even fixing the normalization, an overall phase remains unobservable. Quotienting with respect to these identifications, we get the following double fibration:

\[ \begin{align*}
\mathbb{R}^+ & \hookrightarrow \mathcal{H}_0 = \mathcal{H} - \{0\} \\
U(1) & \hookrightarrow S^{2n-1} \\
P(\mathcal{H}) & \hookrightarrow \mathcal{H}
\end{align*} \]  

whose final result is the projective Hilbert space \( \mathcal{P}\mathcal{H} \simeq \mathbb{C}P^{n-1} \).

We have also already mentioned that every equivalence class \([|\psi\rangle]\) can be identified with the rank-one projector: \( \rho_\psi = |\psi\rangle\langle\psi| \), where, as usual the vector \(|\psi\rangle\) is supposed to be normalized. The space of rank-one projectors is usually denoted as \( \mathcal{D}_1(\mathcal{H}) \) and it is then clear that in this way we can identify it with \( \mathcal{P}\mathcal{H} \).

**Remark.** The careful reader might have noticed that it appears that the most natural setting for QM is not primarily the Hilbert space itself but rather \( \mathcal{P}\mathcal{H} = \mathcal{D}_1(\mathcal{H}) \), which is not a vector space. On the other hand, in the first part of these lectures, we have argued that the superposition rule, which leads to interference phenomena, remains one of the fundamental building blocks of QM. Let us therefore see what the superposition principle means in this new context.

We consider, for simplicity, two orthonormal states: \(|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H} \), with the associated projection operators: \( \rho_1 = |\psi_1\rangle\langle\psi_1|, \rho_2 = |\psi_2\rangle\langle\psi_2| \). A general linear superposition is given by the normalized vector: \( |\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle \), where \( c_1, c_2 \in \mathbb{C} \) and \( |c_1|^2 + |c_2|^2 = 1 \). However, the associated projector is \( \rho_\psi = |\psi\rangle\langle\psi| = |c_1|^2 \rho_1 + |c_2|^2 \rho_2 + (c_1 c_2^* \rho_{12} + h.c.) \), where \( \rho_{12} =: |\psi_1\rangle\langle\psi_2| \), and cannot be expressed directly in terms of the initial projectors.

To overcome this difficulty by retaining at the same time the information concerning the relative phase of the coefficients, we adopt the following procedure. First we consider a third, fiducial vector \(|\psi_0\rangle\) with the only requirement that it be not orthogonal neither to \(|\psi_1\rangle\) nor to \(|\psi_2\rangle\). Then it is possible to associate normalized vectors \(|\phi_i\rangle\) with the projectors \( \rho_i \) \((i = 1, 2)\) by setting:

\[ |\phi_i\rangle = \frac{\rho_i|\psi_0\rangle}{\sqrt{\text{Tr}[\rho_i|\psi_0\rangle]}} , \quad i = 1, 2 \]  

Forming now the linear superposition: \( |\phi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle \), one finds easily that the associated projector \( \rho \) is written entirely in terms of rank-one projectors:

\[ \rho = |\phi\rangle\langle\phi| = |c_1|^2 \rho_1 + |c_2|^2 \rho_2 + \frac{c_1 c_2^* \rho_{10} \rho_{20} + h.c.}{\sqrt{\text{Tr}[\rho_{10} \rho_{20} \rho_{00}]}}, \]
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$$= \sum_{i,j=1}^{2} c_i c_j^* \frac{r \rho_i \rho_0 \rho_j}{\sqrt{\text{Tr}[\rho_i \rho_0 \rho_j \rho_0]}}$$  \hfill (103)

Thus, a superposition of rank-one projectors which yields another rank-one projector is possible, but requires the arbitrary choice of the fiducial projector $\rho_0$. This procedure is equivalent to the introduction of a connection on the bundle, usually called the Pancharatnam connection [13]. □.

We will proceed to investigate the properties $PH$, the latter not a linear space but nevertheless has a very rich geometric structure, and show how they arise from the structure of the Hilbert space $\mathcal{H}$.

2.0.7. Geometric structures on the Hilbert space

Let us consider the realification $\mathcal{H}_\mathbb{R}$ of $\mathcal{H}$, together with its tangent bundle $T\mathcal{H}_\mathbb{R} \approx \mathcal{H}_\mathbb{R} \times \mathcal{H}_\mathbb{R}$. Points in $\mathcal{H}_\mathbb{R}$, i.e. in the first factor, will be again denoted by the same Latin letters* and we will use Greek letters for the second factor: e.g. $\psi \in T_x \mathcal{H}_\mathbb{R} \approx \mathcal{H}_\mathbb{R}$. We can associate with every point $x \in \mathcal{H}_\mathbb{R}$ the constant vector field:

$$X_\psi \equiv (x, \psi)$$  \hfill (104)

Then, we can ”promote” $g$ of eq. (92) and $\omega$ of eq. (92) to $(0,2)$ tensor fields by defining:

$$g(x) (X_\psi, X_\phi) \equiv g(\psi, \phi)$$  \hfill (105)

$$\omega(x) (X_\psi, X_\phi) \equiv \omega(\psi, \phi)$$  \hfill (106)

Also, multiplication in $\mathcal{H}$ by the imaginary unit will be represented in $\mathcal{H}_\mathbb{R}$ by the linear operator which sends $(u, v) \rightarrow (-v, u)$, namely by:

$$J = \begin{vmatrix} 0_{n \times n} & -I_{n \times n} \\ I_{n \times n} & 0_{n \times n} \end{vmatrix}$$  \hfill (107)

with the property:

$$J^2 = -I_{2n \times 2n}$$  \hfill (108)

We can promote $J$ to a $(1,1)$ tensor field by setting:

$$J(x) (X_\psi) = (x, J\psi)$$  \hfill (109)

making, as we have already noticed before, the tensorial triple $(g, J, \omega)$ admissible and $\mathcal{H}_\mathbb{R}$ a linear Kähler manifold [18].

Also, since the $(0,2)$-tensors $g$ and $\omega$ define maps from $T\mathcal{H}_\mathbb{R}$ to $T^*\mathcal{H}_\mathbb{R}$ and are both non-degenerate, we can also consider their inverses, i.e. the $(2,0)$ contravariant tensors $G$ (a metric tensor) and $\Lambda$ (a Poisson tensor) mapping $T^*\mathcal{H}_\mathbb{R}$ to $T\mathcal{H}_\mathbb{R}$ and such that

$$G \circ g = \Lambda \circ \omega = I_{T\mathcal{H}_\mathbb{R}}$$  \hfill (110)

* In this context $x = u + iv$ will stand for the real pair $(u, v)$.
$G$ and $\Lambda$ can be used together to define an Hermitian product between any two $\alpha, \beta$ in
the dual $\mathcal{H}_\mathbb{R}^*$ equipped with the dual complex structure $J^*$:

$$\langle \alpha, \beta \rangle_{\mathcal{H}_\mathbb{R}^*} = G(\alpha, \beta) + i\Lambda(\alpha, \beta).$$  \hspace{1cm} (111)

To make these structures more explicit, we may introduce an orthonormal basis
$$\{ e_k \}_{k=1,\ldots,n} \text{ in } \mathcal{H} \text{ and global coordinates } (q^k, p^k) \text{ for } k = 1, \ldots, n \text{ on } \mathcal{H}_\mathbb{R} \text{ defined as }$$

$$\langle e_k, x \rangle = (q^k + ip^k)(x), \forall x \in \mathcal{H}. \text{ Then}^*:\n$$

$$g = dq^k \otimes dq^k + dp^k \otimes dp^k \hspace{1cm} (112)$$

$$\omega = dq^k \otimes dp^k - dp^k \otimes dq^k \hspace{1cm} (113)$$

$$J = dp^k \otimes \frac{\partial}{\partial q^k} - dq^k \otimes \frac{\partial}{\partial p^k} \hspace{1cm} (114)$$

as well as:

$$G = \frac{\partial}{\partial q^k} \otimes \frac{\partial}{\partial q^k} + \frac{\partial}{\partial p^k} \otimes \frac{\partial}{\partial p^k} \hspace{1cm} (115)$$

$$\Lambda = \frac{\partial}{\partial p^k} \otimes \frac{\partial}{\partial q^k} - \frac{\partial}{\partial q^k} \otimes \frac{\partial}{\partial p^k} \hspace{1cm} (116)$$

Introducing complex coordinates: $z^k \equiv q^k + ip^k$, $\bar{z}^k \equiv q^k - ip^k$, we can also write

$$G + i \cdot \Lambda = 4 \frac{\partial}{\partial z^k} \otimes \frac{\partial}{\partial \bar{z}^k}, \hspace{1cm} (117)$$

where$^\dagger$

$$\frac{\partial}{\partial z^k} =: \frac{1}{2} \left( \frac{\partial}{\partial q^k} - i \frac{\partial}{\partial p^k} \right), \hspace{0.5cm} \frac{\partial}{\partial \bar{z}^k} =: \frac{1}{2} \left( \frac{\partial}{\partial q^k} + i \frac{\partial}{\partial p^k} \right). \hspace{1cm} (118)$$

2.0.8. Geometric structures on the space of functions and operators

The geometric structures examined in the previous subsection induces two (non-
associative) real brackets on smooth, real-valued functions on $\mathcal{H}_\mathbb{R}$:

- the (symmetric) Jordan bracket $\{f, h\}_g \equiv G(df, dh)$
- the (antisymmetric) Poisson bracket $\{f, h\}_\omega \equiv \Lambda(df, dh)$

By extending both these brackets to complex functions via complex linearity we obtain
eventually a complex bracket $\{.,.\}_\mathcal{H}$ defined as:

$$\{f, h\}_\mathcal{H} = \langle df, dh \rangle_{\mathcal{H}_\mathbb{R}^*} \equiv \{f, h\}_g + i\{f, h\}_\omega. \hspace{1cm} (119)$$

In coordinates:

$$\{f, h\}_g = 2 \left( \frac{\partial f}{\partial z^k} \frac{\partial h}{\partial \bar{z}^k} + \frac{\partial h}{\partial z^k} \frac{\partial f}{\partial \bar{z}^k} \right), \hspace{0.5cm} \{f, h\}_\omega = \frac{2}{i} \left( \frac{\partial f}{\partial z^k} \frac{\partial h}{\partial z^k} - \frac{\partial h}{\partial z^k} \frac{\partial f}{\partial z^k} \right) \hspace{1cm} (120)$$

$^*$ Summation over repeated indices being understood here and in the rest of the Section.
$^\dagger$ Complex coordinates are employed here and elsewhere in these notes only as a convenient shorthand
or as a stenographic notation. Their use does not mean at all that vector fields on functions that are
holomorphic (or anti-holomorphic).
with also:

\[ J = -i \left( dz^k \otimes \frac{\partial}{\partial z^k} - dz^k \otimes \frac{\partial}{\partial \bar{z}^k} \right) \tag{121} \]

In particular, given any operator \( A \in gl(\mathcal{H}) \) we can define the quadratic function:

\[ f_A(x) = \frac{1}{2} \langle x, Ax \rangle = \frac{1}{2} z^\dagger Az \tag{122} \]

where \( z \) is the column vector \((z_1, ..., z_n)\). It follows immediately from Eq.(120) that, for any \( A, B \in gl(\mathcal{H}) \):

\[ \{ f_A, f_B \}_g = f_{AB + BA} \tag{123} \]
\[ \{ f_A, f_B \}_\omega = f_{AB - BA} \tag{124} \]

So, the Jordan bracket of any two quadratic functions \( f_A \) and \( f_B \) is related to the (commutative) Jordan bracket of \( A \) and \( B \), \([A, B]_+\), defined as:

\[ [A, B]_+ \equiv AB + BA \tag{125} \]

while their Poisson bracket is related to the commutator product (the Lie bracket) \([A, B]_-\) defined as:

\[ [A, B]_- \equiv \frac{1}{i} (AB - BA) \tag{126} \]

In particular, if \( A \) and \( B \) are Hermitian, their Jordan product and their Lie bracket will be Hermitian as well. Hence, the set of Hermitian operators on \( \mathcal{H}_\mathbb{R} \), equipped with the binary operations (125) and (126), becomes a \textit{Lie-Jordan algebra}\(^*\) and the binary product:

\[ (A, B) = \frac{1}{2} \left( [A, B]_+ + i [A, B]_- \right) \tag{127} \]

is an associative product. Indeed: \( (A, B) = AB \).

Coming back to quadratic functions, it is not hard to check that:

\[ \{ f_A, f_B \}_\mathcal{H} = 2f_{AB}, \tag{128} \]

which proves the associativity of the bracket (119) on quadratic functions, i.e.:

\[ \{ \{ f_A, f_B \}_\mathcal{H}, f_C \}_\mathcal{H} = \{ f_A, \{ f_B, f_C \}_\mathcal{H} \}_\mathcal{H} = 4f_{ABC}, \quad \forall A, B, C \in gl(\mathcal{H}). \tag{129} \]

It is clear that \( f_A \) will be a real function iff \( A \) is Hermitian. The Jordan and Poisson brackets will define then a Lie-Jordan algebra structure on the set of real, quadratic functions, and, according to Eq.(129), the bracket \( \{ \cdot, \cdot \}_\mathcal{H} \) will be an associative bracket. We may then define two vector fields, the \textit{gradient} \( \nabla f \) of \( f \) and the \textit{Hamiltonian vector field} \( X_f \) associated with \( f \), defined by:

\[ g(\cdot, \nabla f) = df \quad \text{or} \quad G(\cdot, df) = \nabla f, \]
\[ \omega(\cdot, X_f) = df \quad \text{or} \quad \Lambda(\cdot, df) = X_f \tag{130} \]

\(^*\) We remark parenthetically that all this extends without modifications to the infinite-dimensional case, if we assume: \( A, B \in \mathcal{B}_{sa}(\mathcal{H}) \), the set of bounded self-adjoint operators on the Hilbert space \( \mathcal{H} \).
which allow us also to obtain the Jordan and the Poisson brackets as:

\[
\{f, h\}_g = g(\nabla f, \nabla h) \\
\{f, h\}_\omega = \omega(X_f, X_h)
\]  

(131)  

(132)

Explicitly, in coordinates:

\[
\nabla f = \frac{\partial f}{\partial q^k} \frac{\partial}{\partial q^k} + \frac{\partial f}{\partial p^k} \frac{\partial}{\partial p^k} = 2 \left( \frac{\partial f}{\partial z^k} \frac{\partial}{\partial \bar{z}^k} + \frac{\partial f}{\partial \bar{z}^k} \frac{\partial}{\partial z^k} \right)
\]  

(133)

\[
X_f = \frac{\partial f}{\partial p^k} \frac{\partial}{\partial q^k} - \frac{\partial f}{\partial q^k} \frac{\partial}{\partial p^k} = 2i \left( \frac{\partial f}{\partial z^k} \frac{\partial}{\partial \bar{z}^k} - \frac{\partial f}{\partial \bar{z}^k} \frac{\partial}{\partial z^k} \right)
\]  

(134)

which are such that \( J(\nabla f) = X_f \).

Let us consider more explicitly a linear operator \( A : \mathcal{H} \to \mathcal{H} \) to which we can associate:

i) a quadratic function \( f_A \) as in Eq. (122) and

ii) a vector field: \( X_A : \mathcal{H} \to T\mathcal{H} \) via: 

\( x \mapsto (x, Ax) \). If \( f_A \) is real, i.e. if \( A \) is Hermitian, one finds:

\[
\nabla f_A = X_A
\]  

(135)

and:

\[
X_{f_A} = J(X_A)
\]  

(136)

Indeed, denoting with \((\cdot, \cdot)\) the pairing between vectors and covectors, Eq.(135) holds because:

\[
g(y, X_A(x)) = g(y, Ax) = \frac{1}{2} (\langle y, Ax \rangle_{\mathcal{H}} + \langle Ax, y \rangle_{\mathcal{H}}) = \langle df_A(x), y \rangle
\]  

(137)

while Eq.(136) follows from the second expression in Eq. (?). Thus, we will write:

\[
\nabla f_A = A \text{ and } X_{f_A} = iA
\]  

(138)

**Example: the Dilation and the Phase Vector Fields.** If we consider the identity operator \( \mathbb{I} \), we obtain the *dilation (or Liouville) field*:

\[
\Delta : x \mapsto (x, x)
\]  

(139)

which in real coordinates reads as:

\[
\Delta = q^k \frac{\partial}{\partial q^k} + p^k \frac{\partial}{\partial p^k}
\]  

(140)

which is such that: We can also define the *phase vector field*:

\[
\Gamma = J(\Delta) = p^k \frac{\partial}{\partial q^k} - q^k \frac{\partial}{\partial p^k}
\]  

(141)

that will be considered more explicitly in the next subsection. \( \square \)
2.0.9. The complex projective space

We would like now to discuss in some detail the structure of the complex projective Hilbert space $\mathbb{P}^n$, which, as we have already mentioned, represents the right context to describe a geometric formulation of Quantum Mechanics.

Let us start by recalling that, in the finite dimensional case, $\mathbb{P}^n$ is homeomorphic to $\mathbb{C}P^n$ and it is therefore made up of the equivalence classes of vectors $Z = (Z^0, Z^1, \cdots, Z^n) \in \mathbb{C}^{n+1}$ w.r.t. the equivalence relation $Z \approx \lambda Z$, $\lambda \in \mathbb{C} - \{0\}$. The space $\mathbb{C}P^n$ is endowed with the Fubini-Study metric [1], whose pull-back to $\mathbb{C}^{n+1}$ is given by:

$$g_{FS} = \frac{1}{(Z \cdot \bar{Z})^2} \left[ (Z \cdot \bar{Z})dZ \otimes_S d\bar{Z} - (dZ \cdot \bar{Z}) \otimes_S (Z \cdot d\bar{Z}) \right]$$

(142)

where $Z \cdot \bar{Z} = Z^a \bar{Z}^a$, $dZ \cdot \bar{Z} = dZ^a \bar{Z}^a$, $dZ \otimes_S d\bar{Z} = d\bar{Z}^a d\bar{Z}^a + d\bar{Z}^a dZ^a$, together with the compatible symplectic form:

$$\omega_{FS} = \frac{i}{(Z \cdot \bar{Z})^2} \left[ (Z \cdot \bar{Z})dZ \wedge d\bar{Z} - (dZ \cdot \bar{Z}) \wedge (Z \cdot d\bar{Z}) \right] = d\theta_{FS}$$

(143)

where:

$$\theta_{FS} = \frac{1}{2i} \frac{\bar{Z}dZ - Zd\bar{Z}}{Z \cdot \bar{Z}}$$

(144)

This shows that $\mathbb{P}^n$ is intrinsically a Kähler manifold.

If we now consider the double fibration (101), we can start from $\mathcal{H}$ where we can have the distributions generated by the dilation field $\Delta$ and the phase field $\Gamma = J(\Delta)$, which is involutive as $[\Delta, J(\Delta)] = 0$. Going to the quotient with respect to the foliation associated with this distribution will be a way of generating the ray space $\mathbb{P}^n$ which is independent on any Hermitian structure. Now, contravariant tensorial objects on $\mathcal{H}$ will be projectable if and only if they are left invariant by both $\Delta$ and $\Gamma$, i.e. if they are homogeneous of degree zero and invariant under multiplication of vectors by a phase. Typical quadratic functions that ”pass to the quotient” will be normalized expectation values of the form:

$$\rho_x (A) =: Tr \{ \hat{\rho}_x A \} = \frac{\langle x | A | x \rangle}{\langle x | x \rangle}$$

(145)

with $A$ any linear operator and for any Hermitian structure on $\mathcal{H}$.

Concerning projectability of tensors, the complex structure $J$, being (cfr., e.g., Eq.(121)) homogeneous of degree zero and phase-invariant, will be a projectable tensor, while it is clear that the Jordan and Poisson tensors $G$ and $\Lambda$ defined respectively in Eq.(115) or, for that matter, the complex-valued tensor of Eq.(117) will not be projectable (as they are phase-invariant but homogeneous of degree $-2$). To turn them into projectable objects we will have to multiply them by the ”conformal factor”: $\theta(z) =: z^\dagger z$, thus defining new tensors: $\hat{\Lambda}(z) =: \theta(z) \Lambda(z)$ and similarly for $G$.

The isometries of the Fubini-Study metric on $P(\mathcal{H})$ are just the usual unitary transformations which, in infinitesimal form, are written as:

$$\dot{Z}^a = iA^{ab} Z^b$$

(146)
where $A = [A^{ab}]$ is a Hermitian matrix. Thus a generic Killing vector field has the form:

$$X_A = \dot{Z}^a \partial_{Z^a} - \dot{\bar{Z}}^a \partial_{\bar{Z}^a} = iA^{ab}(Z^b \partial_{Z^a} - \bar{Z}^a \partial_{\bar{Z}^b})$$  \hspace{1cm} (147)$$

Notice that these are exactly the Killing vector fields of $S^{2n+1}$. In particular, for $A = I$ we obtain $X_k = \Gamma$ which is a vertical vector field w.r.t. the Hopf projection $\pi_H : S^{2n+1} \rightarrow \mathbb{C}P^n$.

A straightforward calculation shows that:

$$\omega_{FS}(\cdot, X_A) = \frac{1}{Z \cdot \bar{Z}} [d\bar{Z}^a A^{ab} Z^b + \bar{Z}^a A^{ab} dZ^b] - \frac{\bar{Z}^a A^{ab} Z^b}{(Z \cdot \bar{Z})^2} [dZ^c \bar{Z}^c + Z^c d\bar{Z}^c] = d(i_{X_A} \theta_{FS})$$  \hspace{1cm} (148)$$
i.e. that $X_A$ is the Hamiltonian vector field $X_{f_A}$, $\omega_{FS}(\cdot, X_{f_A}) = df_A$ associated with the (real) quadratic function:

$$f_A = \frac{Z \cdot A \bar{Z}^b}{Z \cdot \bar{Z}^b} = \frac{Z^a A^{ab} \bar{Z}^b}{Z \cdot \bar{Z}^b} = iX_A \theta_{FS}$$  \hspace{1cm} (149)$$

for the Hermitian matrix $A$. In a similar way, one can prove that the gradient vector field $\nabla_{f_A}$, $g_{FS}(\cdot, \nabla_{f_A}) = df_A$, of $f_A$ has the form:

$$\nabla_A = A^{ab}(Z^b \partial_{Z^a} + \bar{Z}^a \partial_{\bar{Z}^b})$$  \hspace{1cm} (150)$$

Now, given any two real quadratic functions $f_A, f_B$ ($A, B$ being Hermitian matrices), their corresponding Hamiltonian vector fields satisfy:

$$\omega_{FS}(X_{f_A}, X_{f_B}) = X_{f_A}(df_B) = f_{AB-BA}$$  \hspace{1cm} (151)$$

Therefore, the Poisson brackets associated with the symplectic form:

$$\{f, g\}_{\omega_{FS}} := -\omega(X_f, X_g)$$  \hspace{1cm} (152)$$

are such that:

$$\{f_A, f_B\}_{\omega_{FS}} = f_{AB-BA}$$  \hspace{1cm} (153)$$

Also:

$$g_{FS}(\nabla_{f_A}, \nabla_{f_B}) = \nabla_{f_A}(df_B) = f_{AB+BA} - f_A \cdot f_B$$  \hspace{1cm} (154)$$

Thus, we can therefore define a Jordan bracket by setting:

$$\{f_A, f_B\}_g := g_{FS}(\nabla_{f_A}, \nabla_{f_B}) + f_A \cdot f_B = f_{AB+BA}$$  \hspace{1cm} (155)$$

One says that a real function on $PH$ is Kählerian iff its Hamiltonian vector field is also Killing. Such functions represent quantum observables. The above calculations show that the space $\mathcal{F}(PH)$ of real quadratic functions on $PH$ consists exactly of all Kählerian functions. To extend this concept to the complex case, one says that a complex valued function on $PH$ is Kählerian iff so its real and imaginary parts. Clearly, any such $f$ is a quadratic function of the form (149) with now $A \in \mathcal{B}(\mathcal{H})$. Also, on the space, $\mathcal{F}^C(PH)$, of Kählerian complex functions one can define both an Hermitian two-form:

$$h(\cdot, \cdot) = g_{FS}(\cdot, \cdot) + i\omega_{FS}(\cdot, \cdot)$$  \hspace{1cm} (156)$$
and and associative bilinear product (star-product) via:
\[ f \star g := f \cdot g + \frac{1}{2} h(df, dg) = \frac{1}{2} \{f, g\} + i\{f, g\}_\omega + f \cdot g \] (157)
under which the space \( \mathcal{F}^C(\mathcal{P}H) \) is closed since \( f_A \star f_B = f_{AB} \), thus obtaining a particular realization of the \( \mathbb{C}^* \)-algebra of bounded operators \( \mathcal{B}(\mathcal{H}) \). \( \square \)

2.0.10. The momentum map

We shall consider now the action of the unitary group \( \mathcal{U}(\mathcal{H}) \) on \( \mathcal{H} \), which is the group of linear transformations that preserve the triple \( (g, \omega, J) \). In the following, we will denote with \( u(\mathcal{H}) \) the Lie algebra of \( \mathcal{U}(\mathcal{H}) \) of anti-Hermitian operators and identify the space of all Hermitian operators with the dual \( u^* \) of \( u(\mathcal{H}) \) via the pairing:
\[ \langle A, T \rangle_u^* = \frac{1}{2} Tr(AB) \] (158)
On \( u^* \) we can define a Lie and a Jordan bracket:
\[ [A, B]_- = \frac{1}{i} (AB - BA), \quad [A, B]_+ = AB + BA. \] (159)
giving to \( u^* \) the structure of a Lie-Jordan algebra [8]. In addition, \( u^* \) is equipped with the scalar product
\[ \langle A, B \rangle_u^* = \frac{1}{2} Tr(AB) \] (160)
which satisfies:
\[ \langle [A, \xi]_-, B \rangle_u^* = \langle A, \langle \xi, B \rangle_- \rangle_u^* \] (161)
\[ \langle [A, \xi]_+, B \rangle_u^* = \langle A, \langle \xi, B \rangle_+ \rangle_u^* \] (162)

With any \( A \in u^* \), we can associate the fundamental vector field \( X_A \) on the Hilbert space corresponding to the element \( \frac{1}{i} A \in u(\mathcal{H}) \) defined by the formula:
\[ \frac{d}{dt} e^{-\frac{i}{t} A}(x)|_{t=0} = iA(x), \quad \forall x \in \mathcal{H} \] (163)
In other words, \( X_A = iA \) which has \( f_A \) as its Hamiltonian function. Thus, for any \( x \in \mathcal{H}_\mathbb{R} \) we obtain a \( \mu(x) \in u^* \) such that:
\[ \langle \mu(x), \frac{1}{i} A \rangle = f_A(x) = \frac{1}{2} \langle x, Ax \rangle_{\mathcal{H}} \] (164)
In such a way we can construct a mapping:
\[ \mu : \mathcal{H}_\mathbb{R} \rightarrow u^* \] (165)
which is called the momentum map.
More explicitly, it follows from Eq.(158) that:
\[ \langle \mu(x), \frac{1}{i} A \rangle = \frac{1}{2} Tr(\mu(x)A) \] (166)
which, when compared with Eq.(164), yields:

$$\mu(x) = |x\rangle\langle x|$$  \hspace{1cm} (167)

We may therefore conclude that the unit sphere in $\mathcal{H}$ can be projected onto $u^*(\mathcal{H})$ in an equivariant way with respect to the coadjoint action of $U(\mathcal{H})$. Also, in finite dimensions, the unit sphere is odd dimensional and the orbit in $u^*(\mathcal{H})$ is symplectic.

With every $A \in u^*(\mathcal{H})$ we can associate, with the by now familiar identification (as with every other linear vector space) of the tangent space at every point of $u^*(\mathcal{H})$ with $u^*(\mathcal{H})$ itself, the linear function (hence a one-form) $\hat{A} : u^*(\mathcal{H}) \to \mathbb{R}$ defined as: $\hat{A} = \langle A, \cdot \rangle_{u^*}$. Then, we can define two contravariant tensors, a symmetric (Jordan) tensor:

$$R(\hat{A}, \hat{B})(\xi) = \langle \xi, [A, B]_+ \rangle_{u^*}$$  \hspace{1cm} (168)

and a Poisson (Konstant-Kirillov-Souriau) tensor:

$$I(\hat{A}, \hat{B})(\xi) = \langle \xi, [A, B]_- \rangle_{u^*}$$  \hspace{1cm} (169)

$(A, B, \xi \in u^*(\mathcal{H}))$. We notice that the quadratic function $f_A$ is the pull-back of $\hat{A}$ via the momentum map since, for all $x \in \mathcal{H}$:

$$\mu^*(\hat{A})(x) = \hat{A} \circ \mu(x) = \langle A, \mu(x) \rangle_{u^*} = \frac{1}{2} \langle x, Ax \rangle_{\mathcal{H}} = f_A(x)$$  \hspace{1cm} (170)

This means also that, if: $\xi = \mu(x)$:

$$(\mu_*)G(\hat{A}, \hat{B})(\xi) = G(df_A, df_B)(x) = \{f_A, f_B\}_g(x) = f_{[A,B]_+}(x) = R(\hat{A}, \hat{B})(\xi)$$  \hspace{1cm} (171)

where the last equality follows from Eq.(123), i.e.:

$$\mu_*G = R$$  \hspace{1cm} (172)

Similarly, by using now Eq.(124), we find:

$$(\mu_*\Lambda)(\hat{A}, \hat{B})(\xi) = \Lambda(df_A, df_B)(x) = \{f_A, f_B\}_\omega(x) = f_{[A,B]_-}(x) = I(\hat{A}, \hat{B})(\xi)$$  \hspace{1cm} (173)

i.e.:

$$\mu_*\Lambda = I$$  \hspace{1cm} (174)

Thus, the momentum map relates the contravariant metric tensor $G$ and the Poisson tensor $\Lambda$ with the corresponding contravariant tensors $R$ and $I$. Together they form the complex tensor:

$$(R + iI)(\hat{A}, \hat{B})(\xi) = 2\langle \xi, AB \rangle_{u^*}$$  \hspace{1cm} (175)

which is related to the Hermitian product on $u^*(\mathcal{H})$.

**Example: the Momentum Map for a two-level system.** Let $\mathcal{H} = \mathbb{C}^2$, the Hilbert space appropriate for a two-level system. We can write any $A \in u^*(\mathbb{C}^2)$ as:

$$A = y^0 I + y \cdot \sigma$$  \hspace{1cm} (176)
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where $\mathbb{I}$ is the $2 \times 2$ identity, $\mathbf{y} \cdot \sigma = y^1 \sigma_1 + y^2 \sigma_2 + y^3 \sigma_3$ and: $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. Every $A \in \mathfrak{u}^*(\mathbb{C}^2)$ is then represented by the (real) "four-vector" $(y_A^0 : \mathbf{y}_A)$ with

$$y_A^k = \frac{1}{2} Tr (\sigma_k A) \; , \; k = 1, 2, 3$$

(177)

With it, we can associate the vector field:

$$\dot{A} = y^0(A) \partial_0 + y^1(A) \partial_1 + y^2(A) \partial_2 + y^3(A) \partial_3$$

(178)

Using then Eq.(176) one proves easily that:

$$AB = (y_A^0 y_B^0 + \mathbf{y}_A \cdot \mathbf{y}_B) \mathbb{I} + (y_A^0 \mathbf{y}_B + y_B^0 \mathbf{y}_A + i \mathbf{y}_A \times \mathbf{y}_B) \cdot \sigma$$

(179)

(with " $\times$ " denoting the standard cross-product of three-vectors) and hence

$$\langle AB \rangle_{\mathfrak{u}^*} = \frac{1}{2} Tr (AB) = y_A^0 y_B^0 + \mathbf{y}_A \cdot \mathbf{y}_B$$

(180)

Moreover:

$$[A, B]^+ = 2 \{ (y_A^0 y_B^0 + \mathbf{y}_A \cdot \mathbf{y}_B) \mathbb{I} + (y_A^0 \mathbf{y}_B + y_B^0 \mathbf{y}_A) \cdot \sigma \}$$

(181)

while:

$$[A, B]^- = 2 \mathbf{y}_A \times \mathbf{y}_B \cdot \sigma$$

(182)

Explicit calculations then show that:

$$R (\xi) = 2 \partial_0 \otimes (\xi^1 \partial_1 + \xi^2 \partial_2 + \xi^3 \partial_3) + 2 (\xi^1 \partial_1 + \xi^2 \partial_2 + \xi^3 \partial_3) \otimes \partial_0 + 2 \xi^0 (\partial_0 \otimes \partial_0 + \partial_1 \otimes \partial_1 + \partial_2 \otimes \partial_2 + \partial_3 \otimes \partial_3)$$

(183)

$$I (\xi) = 2 (\xi^1 \partial_2 \wedge \partial_3 + \xi^2 \partial_3 \wedge \partial_1 + \xi^3 \partial_1 \wedge \partial_2)$$

(184)

We thus find the following tensor:

$$R + i I = 2 [ \partial_0 \otimes y^k \partial_k + y^k \partial_k \otimes \partial_0 + y^0 (\partial_0 \otimes \partial_0 + \partial_k \otimes \partial_k) + i \epsilon_{hkl} y^h \partial_k \otimes \partial_l ] (185)$$

□.

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Bibliography

[2] N. Bohr, Philosophical Magazine 26 (151) (1913) 1
[17] M. Planck, Annalen der Physik 309 (3)(1901) 564