A perspective on Black Hole Horizons from the Quantum Charged Particle

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XXIII International Fall Workshop on Geometry and Physics
Aftermath Week, IEMath Granada
Granada, 8 September 2014

Scheme

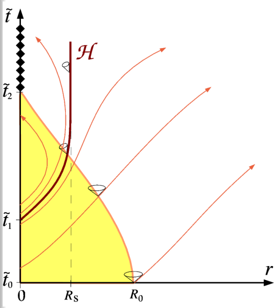
1. Stability of Marginally Outer Trapped Surfaces (MOTS)
2. Motivations and Problem formulation
   - Black Hole Horizon Geometric Inequalities.
   - A Young-Laplace Law for Black Holes.
3. MOTSs and the Quantum Charged Particle
   - MOTSs and Spinors.
   - Spectrum analyticity in the “fine structure constant”.
4. Conclusions and Perspectives

Outline

Establishment's picture of the gravitational collapse

Heuristic chain of Theorems and Conjectures:

1. **Singularity Theorems** (*Theorem*) [Penrose 65, Hawking 67, Hawking & Penrose 70, Hawking & Ellis 73]:
   Sufficient "energy" in a compact region, then light rays converge: notion of *Trapped Surface*.
   Trapped surfaces $\Rightarrow$ spacetime singularity
   (spacetime geodesically incomplete)
2. **(Weak) Cosmic Censorship** (*Conjecture*) [Penrose 69]:
   The singularity should not be visible from a *distant observer*. Preservation of predictability.
   Black Hole region as a region of no-escape.
   Event Horizon as the *Black Hole region* boundary.
3. **Black hole spacetime 'stability'** (*Conjecture*):
   General Relativity gravitational dynamics drive spacetime to stationarity.
4. **BH uniqueness** (*"Theorem"*) [Chruściel et al. 12]:
   The final state of the evolution is a Kerr black hole.



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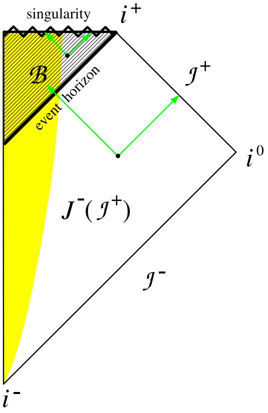
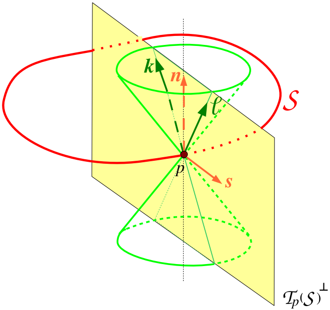


A physical motivation: (future) outer trapping horizons

Let $\mathcal{S}$ be an orientable closed spacelike (codimension 2) surface with induced metric q_{ab}:

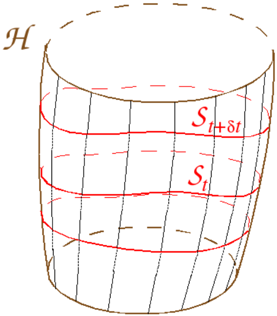
- Normal plane spanned null vectors ℓ^a and k^a
  Normalization: $\ell^a k_a = -1$
  Boost-rescaling freedom:
  $\boxed{\ell'^a = f\ell^a, k'^a = f^{-1}k^a}$, with $f > 0$
- Define the expansions:
  $\theta^{(\ell)} \equiv q^{ab}\nabla_a \ell_b = \frac{1}{\sqrt{q}}\mathcal{L}_\ell \sqrt{q}$
  $\theta^{(k)} \equiv q^{ab}\nabla_a k_b = \frac{1}{\sqrt{q}}\mathcal{L}_k \sqrt{q}$
- Marginally Outer Trapped Surface (MOTS):
  $\theta^{(\ell)} = 0$



A physical motivation: (future) outer trapping horizons

Trapping Horizon [Hayward 94]:

A **Trapping Horizon** is (the closure of) a hypersurface $\mathcal{H}$ foliated by closed marginal surfaces:
$\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$, with $\theta^{(\ell)} =|_{\mathcal{S}_t} 0$.



- Sign of $\theta^{(k)}$: controls if singularity occurs in the *future* or in the *past*.
- Sign of $\delta_k\theta^{(\ell)}$: controls the (local) *outer*- or *inner* character of $\mathcal{H}$.

Trapping Horizons are called:

i) **Future**: if $\theta^{(k)} < 0$. **Past**: if $\theta^{(k)} > 0$.

ii) **Outer**: if $\boxed{\delta_k\theta^{(\ell)} < 0}$. **Inner**: if $\delta_k\theta^{(\ell)} > 0$.

MOTS Stability: Stability operator

Stably outermost MOTS in a normal direction v^a [Andersson, Mars & Simon 05, 08]

Definition. *Given a closed orientable marginally outer trapped surface* $\mathcal{S}$ *and a vector* v^a *orthogonal to it, we will refer to* $\mathcal{S}$ *as* stably outermost with respect to the direction v^a *iff there exists a function* $\psi > 0$ *on* $\mathcal{S}$ *such that the variation of* $\theta^{(\ell)}$ *with respect to* ψv^a *fulfills the condition*

$$\delta_{\psi v}\theta^{(\ell)} \geq 0$$

MOTS Stability operator

The MOTS stability operator along a normal direction v^a to $\mathcal{S}$ is given by:

$$L_v\psi \equiv \delta_{\psi v}\theta^{(\ell)}$$

In particular, for $v^a = -k^a$, we write $L_{\mathcal{S}} \equiv L_{-k}$:

$$L_{\mathcal{S}}\psi = \left[\Delta + 2\Omega^{(\ell)}_a D^a - \left(\Omega^{(\ell)}_a \Omega^{(\ell)a} - D^a\Omega^{(\ell)}_a - \frac{1}{2}R_{\mathcal{S}} + G_{ab}k^a\ell^b\right)\right]\psi$$

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Imposing $G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$, let us define also the operator $L^*_{\mathcal{S}}$ associated to matter:

$$L^*_{\mathcal{S}}\psi = \left[-\Delta + 2\Omega^{(\ell)}_a D^a - \left(\Omega^{(\ell)}_a \Omega^{(\ell)a} - D^a\Omega^{(\ell)}_a - \frac{1}{2}R_{\mathcal{S}} + 8\pi\, T_{ab}k^a\ell^b\right)\right]\psi$$

MOTS Stability: Spectral characterization

Principal eigenvalue of L_v

Let us consider the eigenvalue problem:

$$L_v\phi = \delta_{\phi v}\theta^{(\ell)} = \lambda\, \phi$$

Definition. *The eigenvalue λ_o with the smallest real part, is called the* **principal eigenvalue**.

Spectral characterization of MOTS stability [Andersson, Mars & Simon 05, 08]

Lemma. *The principal eigenvalue λ_o of L_v is real. Moreover, the corresponding principal eigenfunction ϕ_o is either everywhere positive or everywhere negative.*

Lemma. *Let $\mathcal{S}$ be a MOTS and let λ_o be the principal eigenvalue of the corresponding operator $L_{\mathcal{S}} = L_{-k}$. Then $\mathcal{S}$ is stably outermost iff:*

$$\lambda_o \geq 0$$

Remark: notion of *generic Non-Expanding-Horizon* [Ashtekar, Beetle & Lewandowski 02]

A non-extremal NEH (see later) is *generic* if $L_{\mathcal{S}} = L_{-k}$ has trivial kernel.

Rayleigh-Ritz-like characterization of λ_o

Theorem [Andersson, Mars & Simon 08]

The principal eigenvalue λ_o can be written as

$$\lambda_o = \inf_{\psi>0} \int_{\mathcal{S}} \left[|D\psi|^2 + \left(\frac{1}{2}{}^2R - G_{ab}k^a\ell^b \right) \psi^2 - |D\omega_\psi + z|^2 \psi^2 \right] dA$$

where $\Omega_a^{(\ell)} = z_a + D_a f$ (with $D_a z^a = 0$, for any closed Riemannian $\mathcal{S}$), $\int_{\mathcal{S}} \psi^2 dA = 1$ and ω_ψ satisfies, for a given $\psi > 0$

$$-\Delta\omega_\psi - \frac{2}{\psi} D_a\psi D^a\omega_\psi = \frac{2}{\psi} z^a D_a\psi$$

Outline

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MOTS Stability and Horizon Area Inequalities

Stationary Black Holes cannot rotate arbitrarily fast

Their angular momentum J is bounded by their mass M:

$$\boxed{J \leq M^2}$$

Is there a (quasi-local) dynamical version of this bound?

Requirements on the closed surfaces $\mathcal{S}$ ("sections" of the Black Hole horizon)

Need of:

i) **Geometric characterization** of $\mathcal{S}$ in a Black Hole spacetime.

ii) **Stability condition**.

MOTS Stability and Horizon Area Inequalities

Integral characterization of MOTS stability

Lemma. *Given an axisymmetric closed marginally trapped surface* $\mathcal{S}$ *(with axial Killing* η^a *on* $\mathcal{S}$*) satisfying the stably outermost condition for an axisymmetric* $X^a = \gamma\ell^a - \psi k^a$*, then for all axisymmetric* α *it holds*

$$\int_{\mathcal{S}} \left[D_a\alpha D^a\alpha + \frac{1}{2}\alpha^2\ {}^2\!R\right] dS \geq$$

$$\int_{\mathcal{S}} \left[\alpha^2\Omega^{(\eta)}_a\Omega^{(\eta)a} + \alpha\beta\sigma^{(\ell)}_{ab}\sigma^{(\ell)ab} + G_{ab}\alpha\ell^a(\alpha k^b + \beta\ell^b)\right] dS\ ,$$

where $\beta = \alpha\gamma/\psi$.

Remarks:

- The inequality can be obtained from the **Rayleigh-Ritz-like** characterization of the principal eigenvalue λ_o.
- Spacetime expression in which the positivity of the rhs is guaranteed by energy conditions: form of a **"energy-flux inequality"**.

Horizon Geometric Inequalities

Area-angular momentum inequality for outermost stably MOTS

Theorem [JLJ, Reiris & Dain 11]. *Given an axisymmetric closed marginally trapped surface* $\mathcal{S}$ *satisfying the (axisymmetry-compatible) spacetime stably outermost condition, in a spacetime with non-negative cosmological constant and fulfilling the dominant energy condition, it holds the inequality*

$$\boxed{A \geq 8\pi |J|}$$

where A *and* $J = \frac{1}{8\pi}\int_{\mathcal{S}} \Omega_a^{(\ell)} \eta^a dS$ *are the area and (Komar) angular momentum of* $\mathcal{S}$*. If equality holds, then i) the geometry of* $\mathcal{S}$ *is that* **extreme Kerr** *throat sphere, and ii) if* X^a *is spacelike then* $\mathcal{S}$ *is a section of a* non-expanding horizon.

Last step in a series of works along two lines of research: [Ansorg & Pfister 08, Ansorg, Cederbaum & Hennig 08, 10, 11] & [Dain 10, Aceña, Dain & Gabach-Clément 11; Dain & Reiris 11]

Clarification of the relation (variational problem): [Chruściel et al. 11; Mars 12; Gabach-Clément & JLJ 12].

Non-existence of equilibrium aligned rotating BHs: [Neugebauer & Hennig 09, 11, 12].

Horizon Geometric Inequalities

Area-angular momentum-Charge inequality for outermost stably MOTS

Theorem [Gabach-Clément, JLJ & Reiris 12]. *Given an axisymmetric closed marginally trapped surface $\mathcal{S}$ satisfying the (axisymmetry-compatible) spacetime stably outermost condition, in a spacetime with non-negative cosmological constant and fulfilling the dominant energy condition, it holds the inequality*

$$\boxed{(A/(4\pi))^2 \geq (2J)^2 + (Q_{\rm E}^2 + Q_{\rm M}^2)^2}$$

where A is the area of $\mathcal{S}$ and:

$$J = J_{\rm K} + J_{\rm EM} = \frac{1}{8\pi}\int_{\mathcal{S}} \Omega^{(\ell)}_a \eta^a dS + \frac{1}{4\pi}\int_{\mathcal{S}} (A_a\eta^a) F_{ab}\ell^a k^b dS$$

$$Q_{\rm E} = \frac{1}{4\pi}\int_{\mathcal{S}} F_{ab}\ell^a k^b dS \quad , \quad Q_{\rm M} = \frac{1}{4\pi}\int_{\mathcal{S}} {}^*F_{ab}\ell^a k^b dS.$$

If equality holds, then i) the geometry of $\mathcal{S}$ is that extreme Kerr-Newman throat sphere, and ii) if X^a is spacelike then $\mathcal{S}$ is a section of a non-expanding horizon*.*

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Some remarks on the Area Geometric Inequalities

Cosmological constant Λ shift of the princioal eigenvalue λ_o

- Need of solving **Variational Problem when Symmetry** is present.
- Area-Charge inequality [Dain, JLJ & Reiris 11]: $A \geq 4\pi\left(Q_{\mathrm{E}}^2 + Q_{\mathrm{M}}^2\right)$.
  No need of symmetry requirements. No need of variational principle.
- Extension to include the Cosmological constant Λ [Simon 11]:

$$\lambda_o^* A^2 - 4\pi(1-g)A + (4\pi)^2 \sum_i Q_i^2 \leq 0$$

with $\lambda_o^* = \lambda_o + \Lambda$, where $L_{\mathcal{S}}\phi_o = \lambda_o\phi_o \quad , \quad L_{\mathcal{S}}^*\phi_o = \lambda_o^*\phi_o$.

Principal eigenvalue λ_o acts as a Cosmological Constant Λ

Outline

Young-Laplace law for equilibrium soap bubbles

Young-Laplace Law

In equilibrium, each point at the interface between two fluids satisfies

$$\Delta p = p_{\rm inn} - p_{\rm out} = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) ,$$

where:

- $p_{\rm inn}$ and $p_{\rm out}$ are the pressures of the "inner" and "outer" fluids.
- γ is the *surface tension* at the interface: $\delta E = \gamma \, \delta A$.
- R_1 and R_2 are the principal radii of curvature: $H \equiv \frac{1}{R_1} + \frac{1}{R_2}$ is the mean curvature.

Principal eigenvalue for stationary axisymmetric horizons

Principal eigenvalue λ_o for equilibrium axisymmetric BH horizons

Theorem [Reiris 13]. *Given an axisymmetric Isolated Horizon* $\mathcal{H}$ **in arbitrary dimensions (*)** *with null generator* ℓ^a *and non-affinity coefficient* $\kappa^{(\ell)}$*:*

- *There exists an (axisymmetric) foliation of* $\mathcal{H} = \bigcup_t \mathcal{S}_t^{\rm YL}$ *with constant ingoing expansion* $\theta^{(k)}$.
- *The principal eigenvalue* λ_o *evaluated in these sections is*

$$\boxed{\lambda_o = -\kappa^{(\ell)}\theta^{(k)}}$$

- *The principal eigenfunction* ϕ_o *is given by*

$$\phi_o = e^{2\chi}$$

with $\Omega_a^{(\ell)}|_{S^{\rm YL}} = D_a\chi + z_a$, *where* $D^a z_a = 0$.

Remark: In an IH, the principal eigenvalue λ_o does not depend on the section [Mars 12].

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$$\boxed{\lambda_o/(8\pi) = \kappa^{(\ell)}/(8\pi)\ (-\theta^{(k)})}$$

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Comparison to the Young-Laplace law I

First factor in the right-hand-side: $\kappa^{(\ell)}$ as a surface tension

i) **Thermodynamical perspective (energy density):**
Horizon fluid analogy [Smarr 73] based on BH *1st Law* ($\delta M = \frac{\kappa^{(\ell)}}{8\pi}\delta A + \Omega \delta J$), and Smarr formula for the BH mass ($M = 2\frac{\kappa^{(\ell)}}{8\pi}A + 2\Omega J$), leads to:

$$\gamma_{BH} = \kappa^{(\ell)}/(8\pi)$$

ii) **Mechanical perspective (2D-"pressure"):**
Horizon evolution equations for $\theta^{(\ell)}$ and $\Omega_a^{(\ell)}$...

$$\begin{aligned}
\delta_\ell \theta^{(\ell)} - \kappa^{(\ell)}\theta^{(\ell)} &= -\frac{1}{2}{\theta^{(\ell)}}^2 - \sigma^{(\ell)}_{ab}{\sigma^{(\ell)}}^{ab} - 8\pi T_{ab}\ell^a\ell^b \ , \\
\delta_\ell \Omega^{(\ell)}_a + \theta^{(\ell)}\,\Omega^{(\ell)}_a &= {}^2D_a\left(\kappa^{(\ell)} + \frac{\theta^{(\ell)}}{2}\right) - {}^2D_c{\sigma^{(\ell)}}^c{}_a + 8\pi T_{cd}\,\ell^c q^d{}_a
\end{aligned}$$

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ii) **Mechanical perspective (2D-"pressure"):**
... as energy/momentum eqs in the "membrane paradigm" [Damour 78, 79; Znajek 77, 28; Price & Thorne 86...], under $\varepsilon \equiv -\theta^{(\ell)}/8\pi$ and $\pi_a \equiv -\Omega_a^{(\ell)}/(8\pi)$

$$\delta_\ell \varepsilon + \theta^{(\ell)}\varepsilon = -\left(\frac{\kappa^{(\ell)}}{8\pi}\right)\theta^{(\ell)} - \frac{1}{16\pi}(\theta^{(\ell)})^2 + \sigma^{(\ell)}_{cd}\left(\frac{\sigma^{(\ell)\,cd}}{8\pi}\right) + T_{ab}\ell^a\ell^b$$

$$\delta_\ell \pi_a + \theta^{(\ell)}\pi_a = -{}^2D_a\left(\frac{\kappa^{(\ell)}}{8\pi}\right) + {}^2D^c\left(\frac{\sigma^{(\ell)}_{ca}}{8\pi}\right) - {}^2D_a\left(\frac{\theta^{(\ell)}}{16\pi}\right) - q^c{}_a T_{cd}\ell^d$$

Comparison to the Young-Laplace law II

Second factor in the right-hand-side: : $-\theta^{(k)}$ is a mean curvature H

Considering $\mathcal{S}$ embedded in an appropriately boosted 3-slice Σ so that $\ell^a = n^a + s^a$ and $k^a = (n^a - s^a)/2$. Then

$$\left.\begin{array}{rcl} \theta^{(\ell)} & = & q^{ab}\left(\nabla_a n_b + \nabla_a s_b\right) = P + H \\ \theta^{(k)} & = & \frac{1}{2}q^{ab}\left(\nabla_a n_b - \nabla_a s_b\right) = \frac{1}{2}(P - H) \end{array}\right\} \Longrightarrow H = -\theta^{(k)} + \frac{1}{2}\theta^{(\ell)}$$

For MOTS $\theta^{(\ell)} = 0$, so that: $\boxed{-\theta^{(k)} = H}$

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Interpretation proposal: $\lambda_0/(8\pi)$ as a pressure difference

Matching of $\lambda_o/(8\pi) = \kappa^{(\ell)}/(8\pi)\ (-\theta^{(k)})$ **with the form of a Young-Laplace law** achieved, if $\lambda_o/(8\pi)$ is *formally* identified with a pressure difference:

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Does this make sense?

Principal eigenvalue λ_o and pressure

The principal eigenvalue λ_o as a *pressure*

i) λ_o has the same nature as the Cosmological Constant: Λ "shifts" the eigenvalues

$$L_{\mathcal{S}}\phi = \lambda\psi \quad , \quad L^*_{\mathcal{S}}\phi = \lambda^*\phi \quad \Longrightarrow \quad \boxed{\lambda^* = \lambda + \Lambda}$$

ii) The Cosmological Constant Λ **IS** a pressure: $\boxed{P_{\mathrm{cosm}} = -\Lambda/(8\pi)}$

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Stability operator as a "Pressure Operator"

Consider the *evolution* vector h^a on the horizon $\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$, written as $h^a = \ell^a - Ck^a$. Then the trapping horizon condition, $\delta_h\theta^{(\ell)} = 0$ writes

$$\delta_{\ell - Ck}\theta^{(\ell)} = 0 \quad \Longleftrightarrow \quad \delta_{-Ck}\theta^{(\ell)} = -\delta_\ell\theta^{(\ell)}$$

$$\boxed{L_{\mathcal{S}}C = {\sigma^{(\ell)}}_{ab}{\sigma^{(\ell)}}^{ab} + 8\pi T_{ab}\ell^a\ell^b}$$

The rhs fixes the physical dimensions the stability operator:

$$[L_{\mathcal{S}}/(8\pi)] = \mathrm{Energy}\cdot\mathrm{Time}^{-1}\cdot\mathrm{Area}^{-1} = \mathrm{Pressure}$$

MOTS-stability from a BH Young-Laplace law perspective

BH Young-Laplace "law" [JLJ 13]

For stationary axisymmetric IHs, there exists a foliation in which the identifications

$$\begin{aligned} \kappa^{(\ell)}/(8\pi) &\rightarrow \gamma_{BH} \\ -\theta^{(k)} &\rightarrow H = (1/R_1 + 1/R_2) \\ \lambda_o/(8\pi) &\rightarrow \Delta p = p_{\rm inn} - p_{\rm out} \end{aligned}$$

permit to recast the principal eigenvalue in the form of a **Young-Laplace law**:

$$\lambda_o/(8\pi) = \kappa^{(\ell)}/(8\pi)\,(-\theta^{(k)}) \iff \Delta p = p_{\rm inn} - p_{\rm out} = \gamma_{\rm BH} H$$

In this view, **MOTS-stability ($\lambda_o \geq 0$) is interpreted as the result of an increase in the pressure of the trapped region**.

Problem Proposal: Full spectral analysis of $L_{\mathcal{S}}$

- The **whole Stability Operator** $L_{\mathcal{S}}$ argued to represent a "Pressure Operator", $[L_{\mathcal{S}}/(8\pi)] = \mathrm{Presure}$.
- Beyond the principal eigenvalue λ_o, interest in the **full spectrum** of $L_{\mathcal{S}}$.
- Complex λ_n's, *might* play a role in the analysis of instabilities (characteristic frequencies and timescales): $L_{\mathcal{S}}\phi_n = \lambda_n\phi_n = [\mathrm{Re}(\lambda_n) + i\mathrm{Im}(\lambda_n)]\,\phi_n$
- In particular, $L_{\mathcal{S}}$ not self-adjoint for not vanishing $2\Omega^{(\ell)}_a D^a$ term, i.e. with rotation (rotational instabilities, **superradiance** (?)...):

$$J = \frac{1}{8\pi}\int_{\mathcal{S}} \Omega^{(\ell)}_a \eta^a dS$$

Proposal: *"Can one hear the stability of a Black Hole horizon?"* [JLJ 13]

- Systematic **full spectrum analysis of** $L_{\mathcal{S}}$ as a probe into BH horizon (in)stability (kind of "inverse spectral problem" [cf. problem by Kac 66]).
- **Semiclassical tools** for qualitative aspects of the $L_{\mathcal{S}}$ spectrum? [e.g. Berry, Nonnenmacher...].

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   - MOTSs and Spinors.
   - Spectrum analyticity in the “fine structure constant”.

4. Conclusions and Perspectives

From MOTS-stability to the quantum charged particle

Based on [JLJ 14, in preparation].

MOTS-stability operator

The operator $L_{\mathcal{S}}$ is not self-adjoint:

$$L_{\mathcal{S}} = -\Delta + 2{\Omega^{(\ell)}}^a D_a - \left(\Omega^{(\ell)}_a {\Omega^{(\ell)}}^a - D^a \Omega^{(\ell)}_a - \frac{1}{2} R_{\mathcal{S}} + G_{ab} k^a \ell^b \right)$$

Structural similarity with the quantum charged particle

$$\boxed{\Omega^{(\ell)}_a \to \frac{iq}{\hbar c} A_a \quad , \quad R_{\mathcal{S}} \to \frac{4mq}{\hbar^2} \phi \quad , \quad G_{ab} k^a \ell^b \to -\frac{2m}{\hbar^2} V}$$

the MOTS-stability operator becomes $\frac{\hbar^2}{2m} L_{\mathcal{S}} \to \hat{H}$ where

$$\hat{H} \quad = \quad -\frac{\hbar^2}{2m}\Delta + \frac{i\hbar q}{mc} A^a D_a + \frac{i\hbar q}{2mc} D_a A^a + \frac{q^2}{2mc^2} A_a A^a + q\phi + V$$

From MOTS-stability to the quantum charged particle

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The operator $L_{\mathcal{S}}$ is not self-adjoint:

$$\left[-\left(D-\Omega^{(\ell)}\right)^2+\frac{1}{2}R_{\mathcal{S}}-G_{ab}k^a\ell^b\right]\psi=\lambda\psi$$

Structural similarity with the quantum charged particle

$$\hat{H}=\frac{1}{2m}\left(-i\hbar D-\frac{q}{c}A\right)^2+q\phi+V$$

Hamiltonian operator of a **non-relativistic (spin-0) quantum particle of mass** m **and charge** q moving on $\mathcal{S}$ under a magnetic and electric fields with vector and scalar potentials given by A^a and ϕ, and an additional external potential V.

Gauge freedom in the MOTS-stability problem I

Boost/null rescaling freedom

Under the normalization $\ell^a k_a = -1$, we have the freedom
$$\ell'^a = f\ell^a \quad , \quad k'^a = f^{-1}k^a$$
with $f > 0$ to preserve time orientation.

MOTS condition preserved

The expansion rescales

$$\theta^{(\ell')} \quad = \quad f\theta^{(\ell)}$$

so that $\theta^{(\ell)} = 0$ is invariant.

Hájiček or rotation 1-form $\Omega_a^{(\ell)}$ transformation

The form $\Omega_a^{(\ell)} = -k^c q^d{}_a \nabla_d \ell_c$, transforms as a connection

$$\Omega_a^{(\ell')} \quad = \quad \Omega_a^{(\ell)} + D_a(\ln f)$$

Gauge freedom in the MOTS-stability problem II

Interpretation of the Hájíček form

Geometrically is indeed a connection in the normal plane:

$$D_a^{\perp}(\alpha \ell_b + \beta k_b) = (D_a \alpha + \Omega_a^{(\ell)} \alpha)\ell_b + (D_a \beta - \Omega_a^{(\ell)} \beta) k_b$$

Physically is a kind of angular momentum density: $J[\phi] = \frac{1}{8\pi} \int_{\mathcal{S}} \Omega_a^{(\ell)} \phi^a dA$

Spectral problem invariance: analogy to Schrödinger equation $U(1)$-invariance

Under the gauge transformations $\ell'^a = f\ell^a, k'^a = f^{-1}k^a, \Omega_a^{(\ell')} = \Omega_a^{(\ell)} + D_a(\ln f)$ the MOTS-stability operator transforms as:

$$(L_{\mathcal{S}})'\psi = f L_{\mathcal{S}}(f^{-1}\psi)$$

Consider the eigenfunction transformation: $\boxed{\psi' = f\psi}$.
Then the spectral problem (stationary Schrödinger equation) is invariant:

$$L_{\mathcal{S}}\psi = \lambda\psi \quad \rightarrow \quad (L_{\mathcal{S}})'\psi' = \lambda\psi'$$

MOTS-stability and quantum charged particle similarities

Spectral problem: $L_{\mathcal{S}} \leftrightarrow \hat{H}$

$L_{\mathcal{S}}\psi = \lambda\psi$ (MOTS) , $\hat{H}\psi = E\psi$ (stationary quantum charged particle)

Abelian gauge symmetry

$$A_a \rightarrow A_a - D_a\sigma \quad , \quad \psi \rightarrow e^{iq\sigma/(c\hbar)}\psi \quad , \quad \text{(quantum charged particle)}$$
$$\Omega_a^{(\ell)} \rightarrow \Omega_a^{(\ell)} - D_a\sigma \quad , \quad \psi \rightarrow f\psi = e^{-\sigma}\psi \quad , \quad \text{(MOTS-spectral problem)}$$

Phase $U(1)$ (charged particle) and rescaling $\mathbb{R}^+$ (MOTS) gauge symmetries.

Operators obtained by "minimal coupling" of the gauge potentials

$$i\hbar\partial_t \rightarrow i\hbar\partial_t - q\phi \quad , \quad -i\hbar D_a \rightarrow -i\hbar D_a - \frac{q}{c}A_a$$
$$D_a \rightarrow D_a - \Omega_a^{(\ell)}$$

MOTS stability and quantum stability

$\lambda_o \geq 0$ and E bounded below

MOTS and negative fine structure constant α

Stable MOTS as quantum particles with negative $\alpha = e^2/(\hbar c)$. Terminology

Setting $\hbar = m = c = 1$ ($\tilde{\phi} \equiv -\phi/e$) and the **fine-structure constant**: $\boxed{\alpha = e^2}$

$$\begin{aligned} \frac{L_{\mathcal{S}}}{2} &= -\frac{1}{2}\Delta + {\Omega^{(\ell)}}^a D_a + \frac{1}{2} D^a \Omega^{(\ell)}_a - \frac{1}{2}\Omega^{(\ell)}_a {\Omega^{(\ell)}}^a + \frac{1}{4} R_{\mathcal{S}} - \frac{1}{2} G_{ab} k^a \ell^b \\ \hat{H} &= -\frac{1}{2}\Delta + i\sqrt{\alpha} A^a D_a + \frac{i\sqrt{\alpha}}{2} D_a A^a + \frac{\alpha}{2} A_a A^a - \alpha\tilde{\phi} + V \end{aligned}$$

- Δ: *kinematical* term,
- $\Omega^{(\ell)}_a$: *magnetic potential* vector.
- $R_{\mathcal{S}}/4$: *electric* potential (actually $R_{\mathcal{S}}/4 \sim \mathrm{Re}(\Psi_2) + ...$).
- $\Omega^{(\ell)}_a {\Omega^{(\ell)}}^a$: **diamagnetic term**.
- $2{\Omega^{(\ell)}}^a D_a$: **paramagnetic term**.
- $G_{ab} k^a \ell^b/2$: *external mechanical* potential.
- $D^a \Omega^{(\ell)}_a$: *gauge-fixing* term.

Outline

First-order operator and Spinors

Interest in a first-order version of stability

- MOTS-stability as a 1st-order condition to be used as an inner boundary condition (Witten's proof of positivity of mass $M \geq 0$, Penrose conjecture...).
- Reduction of the spectral problem to that of a 1st-order operator.
- ...
- **Spinor characterization of MOTS-stability**.

Ideally: Klein-Gordon equation as square of the Dirac equation

$$\begin{aligned} \left[(i\hbar)^2\,\Box + m^2c^2\right]\Psi &= 0 \qquad (-p_0^2 + \vec{p}^{\,2} = -m^2c^2) \\ \left(i\hbar\boldsymbol{\gamma}^i D_i + mc\right)\boldsymbol{\Psi} &= 0 \end{aligned}$$

with Dirac-gamma matrices $\boldsymbol{\gamma}^\mu$

$$\{\boldsymbol{\gamma}^\mu, \boldsymbol{\gamma}^\nu\} = 2\eta^{\mu\nu}\mathbf{1} \quad , \quad \boldsymbol{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Pauli's approach to minimal coupling

Pauli's way to spinors

Note: $\Delta = D^a D_a = (\boldsymbol{\sigma}^i D_i)^2$, with

$$\boldsymbol{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ \boldsymbol{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \ \boldsymbol{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Minimal coupling: two possibilites

Starting from the non-charged particle: $(-i\hbar\boldsymbol{\sigma}^i D_i)^2 \rightarrow \left(\boldsymbol{\sigma}^i(-i\hbar D_i - \frac{q}{a}A_i)\right)^2$ leads to Pauli's equation

$$i\hbar\partial_t\Psi = \left[\frac{1}{2m}(-i\hbar D_a - \frac{q}{c}A_a)^2 - \frac{\hbar q}{2mc}\boldsymbol{\sigma}^i B_i + q\phi + V\right]\boldsymbol{\Psi}$$

for the non-relativistic spin-$\frac{1}{2}$ quantum charged particle with *giromagnetic factor* $g = 2$ (elementary particle) .

MOTS-stability and Pauli operator

Lichnerowicz-Weitzenböck... formula

$$(i\not{D}_A)^2 = \left(i\gamma^a(\mathbf{D}_a - A_a)\right)^2 = -(D_a - A)^2 + \frac{1}{4}R_{\mathcal{S}} + \frac{1}{4}[\gamma^a, \gamma^b]F^A_{ab}$$

with $F^A_{ab} = D_aA_b - D_aA_b + A_aA_b - A_bA_a$.

MOTS case: $A_a \to \Omega^\ell_a$

$$(i\not{D}_\Omega)^2 = \left(i\gamma^a(\mathbf{D}_a - \Omega_a)\right)^2 = -(D - \Omega)^2 + \frac{1}{4}R_{\mathcal{S}} + \frac{1}{4}[\gamma^a, \gamma^b]F^\Omega_{ab}$$

MOTS-stability operator:

$$L_{\mathcal{S}} = (i\not{D}_\Omega)^2 + \frac{1}{4}R_{\mathcal{S}} - \frac{1}{4}[\gamma^a, \gamma^b]F^\Omega_{ab} + G_{ab}k^a\ell^b$$

with $\frac{1}{4}R_{\mathcal{S}}$ electric potential, $-\frac{1}{4}[\gamma^a, \gamma^b]F^\Omega_{ab}$ standard correction to the giromagnetic factor (*non-elementary* particle), $G_{ab}k^a\ell^b$ external potential.

MOTS-stability and Pauli operator

Dimention $d = 2$ case

$$L_{\mathcal{S}} = (i\not{D}_\Omega)^2 + \frac{1}{4}R_{\mathcal{S}} - \frac{1}{4}[\boldsymbol{\sigma}^a, \boldsymbol{\sigma}^b]F^\Omega_{ab} + G_{ab}k^a\ell^b$$

Then $\frac{1}{4}[\boldsymbol{\sigma}^a, \boldsymbol{\sigma}^b]F^\Omega_{ab} = 2i\sigma^3\frac{1}{4}\epsilon^{ab}F^\Omega_{ab}$.

Penrose-Rindler complex scalar $\mathcal{K}$

$$\mathcal{K} = \frac{1}{4}R_{\mathcal{S}} + i\frac{1}{4}\epsilon^{ab}F^\Omega_{ab} \sim \mathrm{Re}(\Psi_2) + i\mathrm{Im}(\Psi_2) + ...$$

where Ψ_2 is one of the (complex) Weyl scalars: components of the **Weyl tensor**, namely the traceless part of the Riemann curvature tensor. In fact, $\mathrm{Re}(\Psi_2)$ is referred as the "Coulombian part" and $\mathrm{Im}(\Psi_2)$ as the "rotation part".

Note: "Isolated Horizon" Multipoles as spherical harmonics of $\mathrm{Re}(\mathcal{K}) \to M_n$ and $\mathrm{Im}(\mathcal{K}) \to J_n$.

MOTS-stability operator as a "non-relativistic limit"

From Dirac to Pauli: "Non-relativistic limit" approach

Pauli equation can be recovered from Dirac equation in the limit $c \to \infty$.

Following the "non-relativistic" strategy $(d = 2)$

Let ϵ be a dimensionless number and L any length dimension. Define

$$\not{D}_{\Omega}^{\epsilon,L} = -\epsilon^{-1}\boldsymbol{\sigma}^a(D_a - \Omega_a) + \sigma^3\left(\frac{\epsilon^{-2}}{L}\right) + L\left(\mathrm{Re}[\mathcal{K}] - 2\,\mathrm{Im}[\mathcal{K}] - G_{ab}k^a\ell^b\right)$$

i) Obtain the eigenvalues: $\not{D}_{\Omega}^{\epsilon,L} = \lambda_{\epsilon,L}\Psi$

ii) Choose the set of eigenvalues whose eigenfunction does not vanish in the limit $\epsilon \to 0$, then the eigenvalues λ to $L_{\mathcal{S}}\psi = \lambda\psi$ can be obtained as

$$\lambda = \lim_{\epsilon \to 0} \lambda_{\epsilon,L}$$

And they are independent of L. The $\epsilon \to 0$ plays the role of the $c \to \infty$ limit.

Outline

A suggestive simple example: "Landau levels"

"Landau levels" for MOTS

Consider S^2 with $\Omega_a^{(\ell)} = \epsilon_a{}^b D_b\omega + D_a\sigma$
Choose the simplest case $\omega = a\cos\theta$:

$$q_{ab} = d\theta^2 + \sin^2\theta d\varphi^2 \ , \ R_{\mathcal{S}} = \frac{2}{r^2} \ , \ \Omega_a^{(\ell)} = (0, a\sin^2\theta) \ , \ \Omega_a^{(\ell)}\Omega^{(\ell)a} = \frac{a^2}{r^2}\sin^2\theta$$

Then $L_{\mathcal{S}}\psi = \lambda\psi$ exactly solved by:

$$\boxed{\lambda = (\lambda_{\ell m} + 1 - a^2) + i2am \quad , \quad \psi = S_{lm}(a, \cos\theta)e^{im\varphi}}$$

where $S_{lm}(a, \cos\theta)$ are the **"prolate"** spheroidal functions, going to the standard spherical harmonics for $a \to 0$: $\lambda_{\ell m} \to \ell(\ell+1)$ and $S_{lm}(a, \cos\theta) \to P_{\ell m}(\cos\theta)$.

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Notice:

This spectral problem can be recast as $L^a_{\mathcal{S}}\psi = \lambda\psi$, with

$$L^a_{\mathcal{S}} = \left[-\Delta + 2a\Omega^{(\ell)a}D_a + aD^a\Omega^{(\ell)}_a - a^2\Omega^{(\ell)}_a\Omega^{(\ell)a} + \frac{1}{2}R_{\mathcal{S}} - G_{ab}k^a\ell^b\right]$$

and $\Omega^{(\ell)}_a = (0, \sin^2\theta)$.

A suggestive simple example: "Landau levels"

MOTS: $\lambda = (\lambda_{\ell m} + 1 - a^2) + i2am \quad , \quad \psi = S_{lm}(a, \cos\theta)e^{im\varphi}$ **(prolate)**

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Relevant remark: complex rotation $a \to ia$

The operator $L_{\mathcal{S}}^{ia}$ is now **self-adjoint** and the problem:

$$L_{\mathcal{S}}^{ia}\psi = \left[-\Delta + 2ia{\Omega^{(\ell)}}^a D_a + iaD^a\Omega_a^{(\ell)} + a^2\Omega_a^{(\ell)}{\Omega^{(\ell)}}^a + \frac{1}{2}R_{\mathcal{S}} - G_{ab}k^a\ell^b\right]\psi = \lambda\psi$$

namely a stationary **quantum-charged particle (QCP)**, has as solutions:

QCP: $\lambda = (\lambda_{\ell m} + 1 + a^2) - 2am \quad , \quad \psi = S_{lm}(ia, \cos\theta)e^{im\varphi}$ **(oblate)**

A suggestive simple example: "Landau levels"

MOTS: $\lambda = (\lambda_{\ell m} + 1 - a^2) + i2am \quad , \quad \psi = S_{lm}(a, \cos\theta)e^{im\varphi}$ **(prolate)**

Relevant remark: complex rotation $a \to ia$

The operator $L_{\mathcal{S}}^{ia}$ is now **self-adjoint** and the problem:

$$L_{\mathcal{S}}^{ia}\psi = \left[-\Delta + 2ia\Omega^{(\ell)^a}D_a + iaD^a\Omega_a^{(\ell)} + a^2\Omega_a^{(\ell)}\Omega^{(\ell)^a} + \frac{1}{2}R_{\mathcal{S}} - G_{ab}k^a\ell^b\right]\psi = \lambda\psi$$

namely a stationary **quantum-charged particle (QCP)**, has as solutions:

QCP: $\lambda = (\lambda_{\ell m} + 1 + a^2) - 2am \quad , \quad \psi = S_{lm}(ia, \cos\theta)e^{im\varphi}$ **(oblate)**

Moral: self-adjoint "trick"

(Landau) MOTS-spectrum problem solved by considering the self-adjoint problem $L_{\mathcal{S}}^{ia}\psi = \lambda\psi$ and making $a \to -ia$ in eigenvalues and eigenfunctions.

MOTS-stability operator and the fine-structure constant α

Analyticity Conjecture

Given an orientable closed surface $\mathcal{S}$ and the one-parameter family of operators

$$\begin{aligned} L_{\mathcal{S}}[\sqrt{\alpha}] &= -\left(D - i\sqrt{\alpha}A\right)^2 - \alpha\tilde{\phi} + V \\ &= -\Delta + 2i\sqrt{\alpha}A^a D_a + i\sqrt{\alpha}D^a A_a + \alpha A_a A^a - \alpha\tilde{\phi} + V \end{aligned}$$

in the (squared-root) of the fine-structure constant $\alpha \equiv \frac{e^2}{\hbar c}$:
the MOTS-spectrum ($\alpha = -1$) can be recovered as an "analytic continuation" of the spectrum in the quantum charged particle spectrum ($\alpha = 1$) self-adjoint problem.

Hopes in a difficult problem in (perturbation) theory of linear operators [Kato 80]

- No boundary conditions (assume topological conditions, if needed).
- Functions q_{ab}, A_a, $\tilde{\phi}$ and V as well-behaved as necessary.
- $L_{\mathcal{S}}[\sqrt{\alpha}]$ is a *self-adjoint holomorphic family of type (A)* [Kato 80]: consequences on analytic continuation of eigenvalues and eingenfunctions...?

From Black Holes to charged particle Quantum Mechanics

If the Analyticity Conjecture proves valid...

The MOTS-stability spectrum problem is "essentially" reduced to that of the self-adjoint problem of the stationary non-relativistic quantum charged particle.

"Inverse" application: ground state energy of the quantum charge particle

A **gauge-invariant** characterization of the ground state E_o is given by

$$E_o = \inf_{\psi>0} \int_{\mathcal{S}} \left(|D\psi|^2 + \left(e\phi + V + |D\omega_\psi + z|^2 \right) \psi^2 \right) dA$$

where $A_a = z_a + D_a f$ (with $D_a z^a = 0$, for any closed Riemannian $\mathcal{S}$), $\int_{\mathcal{S}} \psi^2 dA = 1$ and ω_ψ satisfies, for a given $\psi > 0$

$$-\Delta\omega_\psi - \frac{2}{\psi} D_a\psi D^a\omega_\psi = \frac{2}{\psi} z^a D_a\psi$$

Note: Gauge invariant and the paramagnetic term is recast as a diamagnetic one.

An avenue to semi-classical tools...

Classical Hamiltonian to the $L_{\mathcal{S}}[\sqrt{\alpha}]$ problem

According with the "quantization rule", $p_i \to -iD_i$, (valid in the selfadjoint $\alpha > 0$ case) consider the classical Hamiltonian:

$$H_{\mathrm{cl}}[\sqrt{\alpha}] = (p - \sqrt{\alpha}\Omega^{(\ell)})^2 + \frac{1}{2}R_{\mathcal{S}} - G_{ab}k^a\ell^b$$

Semiclassical approach to the $L_{\mathcal{S}}$ spectral problem

- Considering semiclassical tools analysis (e.g. WKB... [e.g. Berry...]) based on the classical trajectories of the Hamiltonian $H_{\mathrm{cl}}[\sqrt{\alpha}]$ in the phase space.
- On the resulting estimations for eigenvalues and eigenfunctions, make $\sqrt{\alpha} \to -i$.
- **Remark:** analogue to the study of the spectrum of the Laplacian operator, $\Delta_{\mathcal{S}}$, from the properties of geodesics on $\mathcal{S}$.
- Tools employed in Quantum Chaos? [e.g. Berry 80's, Nonnenmacher 10, many others...].
- Spectral zeta function $\zeta_{L_{\mathcal{S}}}(s) = \sum_{\lambda} \frac{1}{\lambda^s}$ [Berry..., Aldana]. Semiclassical approximations... (Anecdote with [Berry 86]...)

Outline

Conclusions and Perspectives

Conclusions

- **Stable MOTS as Quantum Particles with "negative fine-structure constant"**: formal analogy between the MOTS-stability operator of Black Hole apparent horizons and the Hamiltonian of a non-relativistic quantum charged particle.
- **Self-adjoint "shortcut" to the spectral MOTS-problem**: solution of the quantum charged particle problem and analytic extension to negative values of the fine-structure constant. *Transfer of tools from quantum theory*.
- **Semiclassical tools and MOTS-stability**: different potential applications, in particular an avenue to the (very important and very complicate) Kerr case.
- **MOTS and Spinors**: an avenue to the reformulation of MOTS-stability in terms of spinors. Towards a 1st-order formulation. "Non-relativistic" limit.
- **Others**: *Gauge-invariant expression for particle ground state*, MOTS "Aharonov-Bohm effect", signature of quasi-normal modes/superradiance, "BH horizon degrees of freedom" from 2-nd quantization of QCP...?

Conclusions and Perspectives

Perspectives I: An object in the "vertex" of different lines

Seed for a **Research Program**:

- **Analyticity of the spectrum operator**: principal eigenvalue λ_o of Kerr from Quantum Particle ground state E_o.
- **Spectrum statistics** and **Spectral zeta function**: random matrices (Extremal Kerr... and Riemann conjecture for the zeros of the Riemann zeta function?).
- **Semiclassical tools, Dynamical Systems**: "high-eigenvalue" asymptotics and link to quantum billiards.
- **Spinors and Geometric Inequalities**: inner boundary conditions for Penrose conjecture, $A \leq 16\pi M^2$. Link to Sen-Witten connection. Quasi-local gravitational mass. Superradiance...
- **ANR project NOSEVOL**: *"Nonselfadjoint operators, semiclassical analysis and evolution equations"*.
  GDR DYNQUA: "Quantum Dynamics".

Conclusions and Perspectives

Perspectives II: An object in the "vertex" of different lines

- **Higher (BH) dimensions**. Richer topologies and fields: Hodge-decomposition $\Omega^{(\ell)} = d\alpha + \delta\beta + \gamma$, with γ harmonic.
- **Variational formulations**: Action from Wess-Zumino term in a Chern-Simons action, Ginzburg-Landau functionals (link to Seiberg-Witten theory in the $\dim(\mathcal{S}) = 4$ self-dual case)...
- **(Quantum and) Semiclassical Gravity**: model for inner "fluid" pressure, insight into BH entropy from statistics of the spectrum, Young-Laplace as a "classical-limit test" for quantum inner pressures...
- **Oceanography**: generalized "potential vorticity" q in quasi-geostrophic motions. Physical mechanism for effect of "fast motions" on "slow motions":

$$q = \Delta\psi - \left(\frac{1}{r_{\mathrm{Rossby}}}\right)^2 \psi + \eta \quad , \quad \partial_t q + v \cdot Dq = 0$$

- **Statistical physics of 2-dimensional flows**: Turbulence and large coherent structures (vortices...).

Conclusions and Perspectives

Perspectives II: An object in the "vertex" of different lines

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  $$-\Delta\psi + \frac{R_{\mathcal{S}}}{2}^2\psi = \eta - q \quad \Leftrightarrow \quad L_{\mathcal{S}}\psi = |\sigma^{(\ell)}|^2 + T_{ab}\ell^a\ell^b$$
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- ...

Conclusions and Perspectives

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