Monotonicity of the Hawking energy and time-flat surfaces

Marc Mars

University of Salamanca

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Outline

- Main motivation: the Penrose inequality conjecture.

Consider an asymptotically flat initial data set $(\mathcal{N}, g, \mathcal{K})$.

- Initial data set: Triple $(\mathcal{N}, g, \mathcal{K})$: $(\mathcal{N}, g)$ 3-dim. Riemannian manifold and $\mathcal{K}$ symmetric 2-cov. tensor.
- $(\mathcal{N}, g, \mathcal{K})$ is asymptotically flat if $\mathcal{N} = \mathcal{C} \cup (\mathbb{R}^3 \setminus \overline{B}(R_0))$, $\mathcal{C}$ compact and in Cartesian coordinates in $\mathbb{R}^3 \setminus \overline{B}(R_0)$:

$$g_{ij} - \delta_{ij} = O_{(2)}\left(\frac{1}{R^p}\right), \quad \mathcal{K}_{ij} = O_{(1)}\left(\frac{1}{R^{p+1}}\right), \quad \rho, |J|_g = O\left(\frac{1}{R^q}\right), \quad p > \frac{1}{2}, q > 3.$$

- Dominant Energy Condition (DEC): $\rho \geq |\boldsymbol{J}|_g$ where

$$16\pi\rho := \mathrm{Scal}(g) - |\mathcal{K}|_g^2 + k^2, \qquad 8\pi \boldsymbol{J} := -\mathrm{div}_g(\mathcal{K} - k\, g), \qquad k := \mathrm{tr}_g \mathcal{K}$$

- ADM-energy E_{ADM} and ADM-linear momentum P_{ADM}:

$$E_{ADM} := \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj})\, \nu^i dS_r, \quad P_{i\,ADM} := \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} (\mathcal{K}_{ij} - k\, g_{ij})\, \nu^j dS_r.$$

- Positive Mass Theorem [Schoen & Yau '79]: Under DEC $E_{ADM}^2 - |P_{ADM}|^2 \geq 0$.
- ADM mass: $M_{ADM}^2 := E_{ADM}^2 - |P_{ADM}|^2$.

The Penrose inequality is a strengthening of the positive mass theorem when the initial data has a region of strong gravitational field.

- Let $\Omega \subset \mathcal{N}$ be a domain with smooth boundary $\partial\Omega$. The boundary $\partial\Omega$ is weakly outer trapped if $\mathcal{H}_{\partial\Omega} + \mathrm{tr}_{\partial\Omega}\,\mathcal{K} \leq 0$.
    - $\mathcal{H}_{\partial\Omega}$ mean curvature of $\partial\Omega \hookrightarrow \mathcal{N}$.
- The future trapped region $\mathcal{T}^+_{\mathcal{N}} \subset \mathcal{N}$ is the union of all domains with weakly outer trapped boundary.
- Important result [Andersson & Metzger '07]: $\partial\mathcal{T}^+_{\mathcal{N}}$ is either empty or a smooth surface.

Conjecture (Penrose inequality)

Let $(\mathcal{N}, g, \mathcal{K})$ be a 3-dimensional asymptotically flat initial data set satisfying the dominant energy condition and $\mathcal{T}^+_{\mathcal{N}}$ non-empty. Then, the ADM mass M_{ADM} satisfies

$$16\pi M_{ADM}^2 \geq |S_{\min}(\partial\mathcal{T}^+_{\mathcal{N}})|$$

where $S_{\min}(\partial\mathcal{T}^+_{\mathcal{N}})$ is the infimum of the area of all surfaces enclosing $\mathcal{T}^+_{\mathcal{N}}$. Moreover, if equality holds then $(\Sigma \setminus \mathcal{T}^+_{\mathcal{N}}, g, \mathcal{K})$ can be isometrically embedded into the Kruskal spacetime.

- There are versions of this conjecture also in higher dimensions.

Proven in the Riemannian case ($\mathcal{K} = 0$):

- Dimension $n = 3$ using inverse mean curvature flow [Huisken & Ilmanen '97].
- $n = 3$ using conformal flows [Bray '99]. Extended to $n \leq 7$ [Bray & Lee '09].
- Graphs over hyperplanes in $\mathbb{E}^n$ [Lam '10]. Rigidity [Huang & Wu '12].

Huisken & Ilmanen approach based on the monotonicity of the Hawking mass:
Functional on surfaces Σ embedded in $\mathcal{N}$:

$$M_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(\frac{\chi(\Sigma)}{2} - \frac{1}{16\pi} \int_\Sigma \mathcal{H}_\Sigma^2 d\Sigma \right), \qquad \chi(\Sigma) \text{ Euler characteristic}$$

- Key property [Geroch '73]: $M_H(\Sigma)$ is increasing under inverse mean curvature flow.
  - Flow velocity: $|\xi| = \frac{1}{\mathcal{H}}$ and normal to the surfaces.
- Many issues had to be resolved: find a suitable weak formulation, existence of the weak flow, monotonicity across jumps, limit at infinity, etc.

The Hawking mass has a spacetime version called Hawking energy. Natural to ask whether it still enjoys good monotonicity properties.

- Received attention several years ago + recent developments.

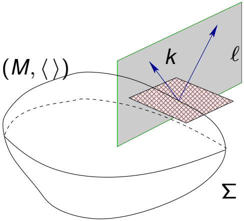
Setup and notation

Spacetime $(M, \langle\,\rangle)$: 4-dimensional Lorentzian manifold (smooth, oriented and time-oriented). Connection ∇.

Surface Σ: Closed two-dimensional, connected, spacelike, embedded submanifold in $(M, \langle\,\rangle)$ (smooth and oriented). Connection ∇^{Σ}.

- At $p \in \Sigma$ decompose $T_pM = T_p\Sigma \oplus T_p^{\perp}\Sigma$.
- Assume trivial normal bundle: $\exists$ two null normal vector fields $k, \ell \in \mathfrak{X}^{\perp}(\Sigma)$ satisfying $\langle k, \ell \rangle = -2$.
- Sign convention for second fundamental form K:

$$\nabla_X Y = \nabla^{\Sigma}_X Y - K(X, Y) \qquad X, Y \in \mathfrak{X}(\Sigma)$$



- Mean curvature vector: $H = \mathrm{tr}_{\Sigma} K$.
- Connection of the normal bundle $\nabla^{\perp}$ and shape operator A_{ζ}

$$\nabla_X \zeta = A_{\zeta}(X) + \nabla^{\perp}_X \zeta \qquad X \in \mathfrak{X}(\Sigma), \quad \zeta \in \mathfrak{X}^{\perp}(\Sigma)$$

- Second fundamental form along a normal direction ζ:

$$K^{\zeta}(X, Y) := \langle K(X, Y), \zeta \rangle$$

For $\eta \in \mathfrak{X}^\perp(\Sigma)$, denote by $\eta^\star$ its Hodge dual vector

$$d\Sigma^\perp(\zeta, \eta^\star) = \langle \zeta, \eta \rangle, \qquad \forall \zeta \in \mathfrak{X}^\perp(\Sigma).$$

Properties:

(i) $\eta^{\star\star} = \eta$, (ii) $\langle \eta, \eta^\star \rangle = 0$, (iii) $\langle \eta^\star, \eta^\star \rangle = -\langle \eta, \eta \rangle$, (iv) $\{\eta, \eta^\star\}$ is basis if and only if $\langle \eta, \eta \rangle \neq 0$ everywhere. Positively oriented if and only if $\langle \eta, \eta \rangle > 0$.

The normal connection $\nabla^\perp$ is uniquely defined in terms of a one-form.

- Given $\nu \in \mathfrak{X}^\perp(\Sigma)$ with $\langle \nu, \nu \rangle = 1$ define the connection one-form along ν as

$$\boldsymbol{\alpha}_{\boldsymbol{\nu}}(X) := \langle \nabla_X \nu, \nu^\star \rangle.$$

- Under a change of basis $\theta \in C^\infty(\Sigma, \mathbb{R})$

$$\nu_\theta = \cosh\theta\, \nu + \sinh\theta\, \nu^\star \qquad \Longrightarrow \qquad \boldsymbol{\alpha}_{\boldsymbol{\nu_\theta}} = \boldsymbol{\alpha}_{\boldsymbol{\nu}} - d\theta.$$

Hawking Energy

- The Hawking energy is a possible measure of the total energy contained in Σ.

$$M_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(\frac{\chi(\Sigma)}{2} - \frac{1}{16\pi} \int_\Sigma \langle H, H \rangle d\Sigma \right).$$

Properties:

- On any connected component of $\partial \mathcal{T}^+_{\mathcal{N}}$: $M_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}$ ✓ (any connected component of $\partial \mathcal{T}^+_{\mathcal{N}}$ is always spherical (Galloway '08))
- The Hawking energy of large spheres $\{\Sigma_r\}$ on any asymptotically flat end $\mathcal{N}^\infty \subset (\mathcal{N}, \gamma, \mathcal{K})$ approaches the ADM energy E_{ADM}. ✓
- The Hawking energy coincides with the Hawking mass if Σ is embedded on a time-symmetric initial data set.
- $M_H(\Sigma)$ is a spacetime object (does not require a $3+1$ decomposition into space and time for its definition).
- Makes sense to analyze spacetime flows.

Does it have any interesting monotonicity properties?

Variation of the Hawking energy along non-null flow [Bray, Hayward, M., Simon '07]

Let $(M, \langle\,\rangle)$ be a spacetime and Σ a surface. Under a flow $\{\Sigma_\lambda\}$ defined by a nowhere null normal field ξ:

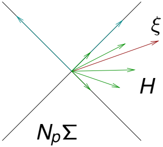
$$\frac{dM_H(\Sigma_\lambda)}{d\lambda} = \frac{1}{8\pi}\sqrt{\frac{|\Sigma_\lambda|}{16\pi}}\int_{\Sigma_\lambda}\Big[\mathrm{Ein}^M(H^\star,\xi^\star) + \Theta^T(H^\star,\xi^\star) + \langle \xi, H^\star\rangle \operatorname{div}_\Sigma \boldsymbol{\alpha}_{\boldsymbol{\nu_\xi}} +$$
$$\left(|\boldsymbol{\alpha}_{\boldsymbol{\nu_\xi}}|^2 + |\nabla^\Sigma\psi|^2\right)\langle \xi, H\rangle + 2\boldsymbol{\alpha}_{\boldsymbol{\nu_\xi}}(\nabla^\Sigma\psi)\langle \xi, H^\star\rangle + (\Delta_\Sigma\psi - \Omega)\left(\langle\xi, H\rangle - \overline{\langle \xi, H\rangle}\right)\Big]\, d\Sigma_\lambda.$$

where

- $e^{2\psi} = |\langle\xi,\xi\rangle|$, $\nu_\xi = e^{-\psi}\xi$ (ξ spacelike) or $\nu_\xi = e^{-\psi}\xi^\star$ (ξ timelike).
- $\overline{U} =$ mean value of U on Σ_λ $\quad \mathring{T} =$ trace-free part of T,
- $\Omega := \frac{1}{2}\mathrm{Scal}(\Sigma) - \frac{1}{4}\langle H, H\rangle$, $\quad \Theta^T(H^\star,\xi^\star)$ bilinear map depending on $\mathring{K}$.

First, second and fourth terms ≥ 0 if

- (i) DEC holds
- (ii) H spacelike or null and ξ points into the same causal quadrant.



- Second factor of last term has no sign. Product made zero by either $\langle \xi, H \rangle =$ const. or

$$\Delta_\Sigma \psi = \Omega + \alpha, \qquad \alpha = \text{ const.}$$

  - $\langle \xi, H \rangle =$ const. is called inverse mean curvature flow (IMCF) Corresponds to the inverse mean curvature flow condition in the time symmetric case.

- Remaining term $\int_{\Sigma_\lambda} \langle \xi, H^\star \rangle \, \mathrm{div}_\Sigma \, \boldsymbol{\alpha}_{\boldsymbol{\nu}_{\boldsymbol{\xi}}} d\Sigma_\lambda \geq 0$ provided

$$\langle \xi, \mathcal{H}^\star \rangle = \text{const.} \qquad \textit{or} \qquad \mathrm{div}_\Sigma \, \boldsymbol{\alpha}_{\boldsymbol{\nu}_{\boldsymbol{\xi}}} = 0$$

  - By analogy $\langle \xi, H^\star \rangle =$ const.: dual inverse mean curvature flow (dual IMCF).

Theorem (Monotonicity of the Hawking energy, [Bray, Hayward, M. Simon '07])

Let $(M, \langle\, \rangle)$ *be a spacetime and* $\{\Sigma_\lambda\}$ *a flow with normal velocity vector* ξ. *Then the Hawking energy* $M_H(\Sigma_\lambda)$ *is monotonically increasing provided:*

(i) *The spacetime satisfies the dominant energy condition.*

(ii) H *is spacelike or null on* Σ_λ, *and* ξ *points into the same causal quadrant.*

(iii) *One of the following four conditions is satisfied:*

(A) *The IMCF and dual IMCF conditions hold.*

(B) *The IMCF condition holds and* ξ *is spacelike and such that*

$$\mathrm{div}_\Sigma\, \boldsymbol{\alpha}_{\boldsymbol{\nu}_\xi} = 0. \qquad (1)$$

(C) *The dual IMCF condition holds and* ξ *is spacelike with norm* $\langle \xi, \xi \rangle = e^{2\psi}$ *satisfying*

$$\Delta_{\Sigma_\lambda} \psi = \frac{1}{2} Scal(\Sigma_\lambda) - \frac{1}{4}\langle H, H \rangle - \alpha(\lambda) \qquad \alpha(\lambda) \in \mathbb{R}, \qquad (2)$$

(D) ξ *is spacelike and (1) and (2) hold.*

- Particular case: H spacelike and $\xi = \frac{H}{\langle H, H \rangle}$ (satisfies $\langle \xi, H \rangle = 1$ (IMCF) and $\langle \xi, H^\star \rangle = 0$ (dual IMCF) [Huisken & Ilmanen '97, Frauendiener '01]

Definition

A surface is admissible if H is spacelike with $H^\star$ future directed.

- For admissible surfaces, the conditions
  - (i) ξ, H belong to the same causal quadrant
  - (ii) inverse mean curvature flow.

  are equivalent to

$$\xi = \frac{1}{\langle H, H \rangle} (H + \beta H^\star) \qquad \beta : \Sigma_\lambda \longrightarrow [-1, 1]$$

- By the first variation of area form

$$\frac{d\Sigma_\lambda}{d\lambda} = \langle \xi, H \rangle d\Sigma_\lambda = d\Sigma_\lambda$$

Definition

A flow of admissible surfaces $\{\Sigma_\lambda\}$ with flow vector

$$\xi = \frac{1}{\langle H, H \rangle} (H + \beta H^\star) \qquad \beta : \Sigma_\lambda \longrightarrow [-1, 1]$$

is called uniformly area expanding flow.

- Recently, Bray and Jauregui have computed the variation of the Hawking energy for uniformly area expanding flows with a different method.

Theorem (Bray, Jauregui '13)

Given a uniformly area expanding flow of admissible surfaces $\{\Sigma_\lambda\}$,

$$\frac{dM_H(\Sigma_\lambda)}{d\lambda} = \frac{1}{8\pi}\sqrt{\frac{|\Sigma_\lambda|}{(16\pi)}}\int_{\Sigma_\lambda}\Big[Ein^M(H^\star,\xi^\star) + \Theta^T(H^\star,\xi^\star) + \left|\frac{\nabla^\Sigma\mathcal{H}}{\mathcal{H}}\right|^2 + 2\beta\,\boldsymbol{\alpha_H}\left(\frac{\nabla^\Sigma\mathcal{H}}{\mathcal{H}}\right) + |\boldsymbol{\alpha_H}|^2 + \beta\,div_\Sigma\,(\boldsymbol{\alpha_H})\Big]\,d\Sigma_\lambda.$$

where $\mathcal{H} := \sqrt{\langle H, H\rangle}$ *and* $\boldsymbol{\alpha_H} := \boldsymbol{\alpha_{\nu_H}}$ *is the connection one-form along* $\nu_H := \frac{1}{\mathcal{H}}H$.

- New condition ensuring monotonicity:

$$\mathrm{div}_\Sigma\,\boldsymbol{\alpha_H} = 0.$$

- Not an extra condition on the flow vector ξ but on the surface Σ itself!.

Fitting the two monotonicity theorems together

- For uniformly expanding flows, the relevant inequality is

$$\int_\Sigma \left(|\boldsymbol{\alpha}_{\nu_\xi}|^2 + |\nabla^\Sigma \psi|^2 - 2\beta \boldsymbol{\alpha}_{\nu_\xi}(\nabla^\Sigma \psi)\rangle - \beta \mathrm{div}_\Sigma (\boldsymbol{\alpha}_{\nu_\xi}) \right) \geq 0.$$

- By the previous results: Fulfilled for arbitrary $|\beta| < 1$ and ψ in two cases: $\mathrm{div}_\Sigma\, \boldsymbol{\alpha}_{\nu_\xi} = 0$ and $\mathrm{div}_\Sigma\, \boldsymbol{\alpha}_{\boldsymbol{H}} = 0$.
- Are there more?

Theorem (Bray, Jauregui, M.)

The Hawking energy is monotonically increasing along a uniformly area expanding flow $\{\Sigma_\lambda\}$*, provided* $\mathrm{div}_\Sigma\,(\alpha_{\nu_\Theta}) = 0$ *where either*

(i) $\nu_\Theta = \cosh(\Theta \circ \beta))\, \nu_\xi + \sinh(\Theta \circ \beta) \nu_\xi^\star$ *or* *(ii)* $\nu_\Theta = \cosh(\Theta \circ \beta)\, \nu_H - \sinh(\Theta \circ \beta)\, \nu_H^\star$,

and $\Theta \in C^\infty((-1, 1), \mathbb{R})$ *is non-decreasing function.*

- $\Theta = 1$ in (i) gives $\mathrm{div}_\Sigma\, \boldsymbol{\alpha}_{\nu_\xi} = 0$, $\Theta = 1$ in (ii) gives the $\mathrm{div}_\Sigma\, \boldsymbol{\alpha}_{\boldsymbol{H}} = 0$ condition.
- The sets (i) and (ii) are disjoint for non-constant β.

Definition

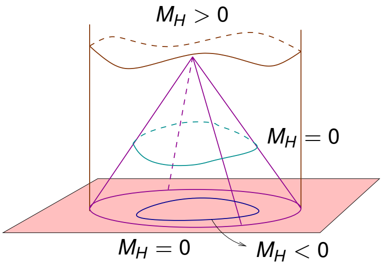
An admissible surface satisfying $\mathrm{div}_\Sigma\, \boldsymbol{\alpha_H} = 0$ is called time-flat.

- The name is intended to capture the idea that the surface has no time "wiggles".

The Hawking energy is known to either overestimate or underestimate the total energy depending on the situation:

In the Minkowski spacetime $\mathbb{M}^{1,3}$:

- Σ embedded in a spacelike hyperplane: $M_H(\Sigma) = 0$ for spheres, < 0 otherwise.
- Σ embedded in the light cone of a point $M_H(\Sigma) = 0$.
- There exist surfaces embedded in a time cylinder $\mathbb{R} \times \{|x| = R\}$ with $M_H(\Sigma) > 0$.



Bounding the ADM energy from below in terms of the Hawking energy needs restritions on the surface $\longrightarrow$ Time-flatness may be a good such restriction.

- All surfaces with spacelike H embedded in a time symmetric hypersurface are time-flat $\longrightarrow$ The IMCF in the time-symmetric case is by time-flat surfaces.

Time-flat surfaces are geometrically natural. Generalize:

- Surfaces with parallel mean curvature $\nabla^{\perp} H = 0$.
- Surfaces with parallel normalized mean curvature $\nabla^{\perp} \nu_H = 0 \quad \Longleftrightarrow \quad \boldsymbol{\alpha_H} = 0$

In order to analyze the time-flat condition it is useful to know its variation formula.

Proposition (Variation of the time-flat condition, [Bray, Jauregui, M.])

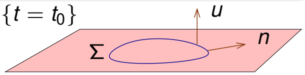
- *On a spacetime $(M, \langle\,\rangle)$ consider a flow of surfaces $\{\Sigma_\lambda\}$ with normal flow vector ξ.*
- *Assume all surfaces to be admissible and define $\nu_H = \frac{H}{\mathcal{H}}$.*

Then, the divergence of the connection one-form $\boldsymbol{\alpha_H}$ satisfies the variation formula

$$\begin{aligned}\mathcal{L}_\xi(div_\Sigma\,\boldsymbol{\alpha_H}) =&\Delta_\Sigma\left[\frac{1}{\mathcal{H}}\left(div_\Sigma(\nabla^\Sigma\xi^{\nu_H^\star}) - div_\Sigma(\xi^{\nu_H}\boldsymbol{\alpha_H}) - \boldsymbol{\alpha_H}(\nabla^\Sigma\xi^{\nu_H}) + \xi^{\nu_H^\star}|\boldsymbol{\alpha_H}|^2\right.\right.\\ &\left.\left. - \mathrm{tr}_\Sigma\left(A_{\nu_H^\star}\circ A_\xi\right) - \langle\nu_H^\star, Riem^M(\xi, X^a)X_a\rangle\right)\right] + \nabla^\Sigma_{\alpha_H}\langle H,\xi\rangle\\ &+ div_\Sigma\left(A_{\nu_H^\star}(\nabla^\Sigma\xi^{\nu_H}) + A_{\nu_H}(\nabla^\Sigma\xi^{\nu_H^\star}) - 3A_\xi(\alpha_H) + \left(Riem^M(\nu_H,\nu_H^\star)\xi\right)^{\parallel}\right)\end{aligned}$$

where $\{X_a\}$ is any basis of $\mathfrak{X}(\Sigma)$ and $\xi = \xi^{\nu_H}\nu_H + \xi^{\nu_H^\star}\nu_H^\star$.

- Specializing to surfaces $\Sigma \hookrightarrow \{t = t_0\} \subset \mathbb{M}^{1,3}$. Decompose $\xi = Fu + Gn$.



$$\mathcal{L}_\xi(\mathrm{div}_\Sigma\,\boldsymbol{\alpha_H}) = \Delta_\Sigma\left(\frac{\Delta_\Sigma F}{\mathcal{H}}\right) + \mathrm{div}_\Sigma\left(A_m(\nabla^\Sigma F)\right).$$

- Recover a resut by [Chen, Wang, Yau '11], [Miao, Tam, Xie '11].

On time-flat surfaces in Minkowski

- The interpretation of time-flat surfaces as surfaces with no "time wiggles" requires some kind of rigidity in the Minkowski spacetime.
- Surfaces with parallel mean curvature vector in Minkowski classified by [Chen & Van der Veken '09] $\longrightarrow$ Compact ones lie necessarily in spacelike hyperplanes.
- Surfaces with parallel normalized mean curvature $\alpha_{\nu_H} = 0$ and spherical topology lie necessarily in space hyperplanes [Chen, Wang, Wang '13].

  Interesting problem to drop the topological restriction.
- Local rigidity result for surfaces in hyperplanes:

Theorem (Chen, Wang, Wang '13)

Any variation of a mean convex surface Σ embedded in a spacelike hyperplane $\mathcal{N}_t$ of $\mathbb{M}^{1,3}$ preserving the time-flat condition must be of the form $\xi = Fu + Gn$ where G is arbitrary and F is the restriction to Σ of a linear function on $\mathcal{N}_t$.

- The deformation moves Σ to another (possibly boosted) spacelike hyperplane.
- The proof consists in studying the solutions of

$$\Delta_\Sigma \left(\frac{\Delta_\Sigma F}{\mathcal{H}} \right) + \mathrm{div}_\Sigma \left(A_n(\nabla^\Sigma F) \right) = 0.$$

- Rigidity for axially symmetric surfaces with positive Gauss curvature.

Theorem (Chen, Wang, Wang '13)

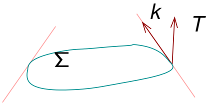
- *Σ an admissible surface embedded in $\mathbb{M}^{1,3}$ with positive Gauss curvature.*
- *Σ a axially symmetric graph over an axially surface $\Sigma_0 \subset \mathcal{N}_t$.*

Then, Σ lies in a totally geodesic hyperplane of $\mathbb{M}^{1,3}$.

- Proof uses positivity of the Wang-Yau mass and properties of critical points of this quasi-local energy.

$\mathrm{div}_\Sigma\, \alpha_H$ has a simple expression in terms of the null extrinsic geometry.

- $\boldsymbol{T} = -dt$, t Minkowskian time, $\tau := t|_\Sigma$.
- k: future directed null normal satisfying $\boldsymbol{T}(k) = -1$.
- $\theta_k := \langle H, k \rangle$, null expansion along k. Negative for admissible Σ.

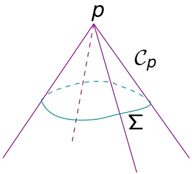


Lemma

Σ admissible surface embedded in $\mathbb{M}^{1,3}$:

$$div_\Sigma\, \boldsymbol{\alpha_H} = \frac{1}{2}\Delta_\Sigma \left[\ln\left(1 + |\nabla^\Sigma \tau|^2 - \frac{2}{\theta_k}\Delta_\Sigma \tau \right) \right] - div_\Sigma \left(A_k(\nabla^\Sigma \tau) \right).$$

- The null extrinsic curvature of surfaces embedded in the null cone of a point is particularly simple.



Theorem

Let $p \in \mathbb{M}^{1,3}$ and $\mathcal{C}_p$ the past or future null cone of p. Any spacelike, admissible and time-flat surface embedded in $\mathcal{C}_p$ must lie in a hyperplane.

- Proof uses properties of the conformal group of the sphere.

Interesting open problem:

Classify compact, spacelike, admissible, time-flat surfaces in the Minkowski spacetime (possibly restricting to graphs over hyperplanes).

- In the case of spacelike curves embedded in $\mathbb{M}^{1,2}$, time-flat $\longrightarrow$ constant torsion.
- Constant torsion curves embedded in spacelike surfaces are flat [Bray, Jauregui '14]

On time-flat surfaces in static spacetime

- We call a spacetime $(M, \langle\,\rangle)$ static if there is a Riemannian manifold (N, g) and a function $V : N \longrightarrow \mathbb{R}^+$ such that

$$M = \mathbb{R} \times N, \qquad \langle\,,\rangle = -V^2 dt^2 + g.$$

- Denoted by (N, g, V). Static Killing $\zeta = \partial_t$.

The general variation formula for $\mathrm{div}_\Sigma\, \boldsymbol{\alpha_H}$ can be specialized to surfaces embedded in constant time slices of static spacetimes

- Curvature terms Riem^M can be computed in terms of V and the geometry of (N, g):

Variation of $\mathrm{div}_\Sigma\, \alpha_H$ in the static case

- Σ: admissible surface embedded $\mathcal{N}_{t_0} = \{t = t_0\}$. Flow vector $\xi|_\Sigma = F\zeta + Gn$ (n unit normal tangent to $\mathcal{N}_{t_0}$)

$$\mathcal{L}_\xi(\mathrm{div}_\Sigma\, \boldsymbol{\alpha_H}) = \Delta_\Sigma \left(\frac{V}{\mathcal{H}} \Delta_\Sigma F + \frac{2}{\mathcal{H}} \langle \nabla^\Sigma F, \nabla^\Sigma V \rangle \right) + \mathrm{div}_\Sigma \left(V A_n(\nabla^\Sigma F) - n(V) \nabla^\Sigma F \right).$$

- Is there a local rigidity result for time-flat surfaces in $\mathcal{N}_{t_0}$?

Theorem (Bray, Jauregui, M.)

- *(N, g, V) static spacetime, Σ an admissible surface embedded in N_{t_0}.*
- *Assume $\Sigma = \partial\Omega \subset N_t$ with Ω compact and the following inequalities hold*

$$Ric^M(X, Y) \geq 0, \quad X, Y \in \mathfrak{X}(N_t), \qquad n(V) \geq \frac{1}{\mathcal{H}V}|\nabla^\Sigma V|^2 \text{ on } \Sigma, \tag{3}$$

Then a variation $F\zeta$ of Σ is such that $\mathcal{L}_{F\zeta}(div_\Sigma \boldsymbol{\alpha_H}) = 0$ if and only if

(i) *$F = F_0$ constant, or*

(ii) *V is constant on Ω and there exists a non-constant function $u : N \mapsto \mathbb{R}$ satisfying $\mathrm{Hess}_N\, u = 0$ and $F = u|_\Sigma$.*

- Case (ii) implies the existence of a unit vector field η in (M, g) with $\nabla^M \eta = 0$ and $\eta \perp \zeta$ (e.g. Minkowski).
- Except for this very symmetric case Σ is locally rigid as a time-flat surface.

Are conditions (3) reasonable?

- $\mathrm{Ric}^M(X, Y) \geq 0$ is an energy-type inequality.
- For round spheres in the Schwarzschild spacetime $n(V) \geq \frac{1}{\mathcal{H}V}|\nabla^\Sigma V|^2$ is $m \geq 0$.

Uniformly area expanding time-flat flow

It makes sense to consider under which conditions a flow of surfaces is uniformly area expanding with all surfaces time-flat.

- Combining uniform expansion with the general variation of $\mathrm{div}_\Sigma\, \boldsymbol{\alpha_H}$:

Theorem (Bray, Jauregui, M.)

Let $\{\Sigma_\lambda\}$ be a flow of admissible surfaces with flow vector $\xi = \frac{1}{\mathcal{H}^2}(H + \beta_\lambda H^\star)$, $\beta_\lambda \in C^\infty(\Sigma_\lambda, \mathbb{R})$. Each surface Σ_λ is time-flat if and only if Σ_0 is time-flat and β_λ satisfy

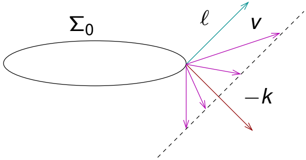
$$\Delta_{\Sigma_\lambda}\left[-\frac{1}{\mathcal{H}}\Delta_\Sigma\left(\frac{\beta_\lambda}{\mathcal{H}}\right) + \frac{\beta_\lambda}{\mathcal{H}^2}\mathcal{A}_1 + \mathcal{A}_2\right] - div_{\Sigma_\lambda}\left[A_{\nu_H}\left(\nabla^\Sigma\left(\frac{\beta_\lambda}{\mathcal{H}}\right)\right) + \frac{\beta_\lambda}{\mathcal{H}}\mathcal{B}_1 + \mathcal{B}_2\right] = 0$$

- $\mathcal{A}_1$, $\mathcal{A}_2$ scalars on Σ_λ and $\mathcal{B}_1$, $\mathcal{B}_1$ vectors on Σ_λ. Explicit expressions in terms of A_{ν_H}, $A_{\nu_H^\star}$ and Riem^M.
- Fourth order elliptic equation for β_λ. Currently analyzing existence.
- No obvious obstruction to existence.

- To address the Penrose inequality we need to start the flow on a surface $\Sigma_0 = \partial \mathcal{T}^+_{\mathcal{N}}$.
- This is a marginally outer trapped surface (MOTS): $\theta_\ell = 0$ where $\theta_\ell := \langle H, \ell \rangle$.
- H is null, so $\mathrm{div}_\Sigma\, \alpha_H = 0$ cannot be imposed on Σ_0

Stability operator:

- Fix a section $v \in \mathfrak{X}^\perp(\Sigma_0)$ nowhere tangent to ℓ.
- Given ℓ, k null basis of $\mathfrak{X}^\perp(\Sigma_0)$ with $\langle \ell, k \rangle = -2$, v uniquely represented by a function $V \in C^\infty(\Sigma_0, \mathbb{R})$



$$v = -\frac{1}{2} k + V \ell.$$

- Given $\psi : \Sigma_0 \longrightarrow \mathbb{R}$, the first order variation of θ_ℓ along ψv. defines the stability operator L_v: $\boldsymbol{L_v \psi \equiv \mathcal{L}_{\psi v} \theta_\ell}$.
- Second order elliptic linear operator, not self-adjoint in general.

L_v has a unique principal eigenvalue λ_v.

- Eigenvalue with smallest real part, always real, eigenspace if one-dimensional, eigenfunction have constant sign.

- Σ_0 is called stable (strictly stable) along v if $\lambda_v \geq 0$ ($\lambda_v > 0$).

Theorem

- *Let $\Sigma_0 = \partial T^+(\mathcal{N})$ be strictly stable and satisfy $\theta_k \leq 0$ and not identically zero.*
- *Consider a spacelike normal flow of surfaces starting at Σ_0 such that $\xi|_{\Sigma_0} = \psi v$.*

Assume that $\{\Sigma_\lambda\}$ for $\lambda > 0$ is a flow of time-flat surfaces. Then

$$L_v(\psi) = -\theta_k e^{-2V_0},$$

$$\Delta_{\Sigma_0} V_0 = -div_\Sigma \, \boldsymbol{s} \qquad \text{where} \qquad \boldsymbol{s}(X) = -\frac{1}{2}\langle \nabla_X \ell, k\rangle.$$

Conversely, any normal flow starting on Σ_0 with initial velocity $\xi = \psi v$, ψ satisfying these equations, is such that

(i) *The flow near Σ_0 is by admissible surfaces.*

(ii) $div_\Sigma \, \alpha_H = O(\lambda^2)$.

- The flow can be started on any strictly stable MOTS with negative expansion θ_k (marginally trapped surface).
- The condition $\theta_k \leq 0, \not\equiv 0$ is used to ensure $\psi > 0$ by the maximum principle for L_v.

Summary and outlook

- The Hawking energy is potentially useful to address the Penrose inequality.
- It satisfies interesting monotonicity properties. Worth investigating the corresponding flows.
- One such flows involves uniform area expansion and time-flat surfaces.
- Time-flat surfaces are geometrically well-motivated and perhaps suitable to keep under control the tendency of the Hawking energy to overestimate the energy.
- For this to be true several results are required, e.g. rigidity type results in Minkowski and static spacetimes.
- Uniformly area expanding time-flat flows can start on any strictly stable marginally trapped (non-minimal) surface.
- Flow equations involve a fourth-order linear elliptic equation. Existence is under current investigation.

Thank you