Notes on the Minicourse:

SUBMANIFOLD THEORY.

1 Introduction to smooth manifolds.

1.1 Smooth manifolds.

Definition 1. *A smooth (or differentiable) manifold of dimension n is a set M and a family of injective mappings $x_\alpha : U_\alpha \subseteq \mathbb{R}^n \longrightarrow M$ of open sets U_α of $\mathbb{R}^n$ into M such that:*

(1) $\cup_\alpha x_\alpha(U_\alpha) = M$,

(2) for any pair α, β, with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset$, the sets $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the mappings $x_\beta^{-1} \circ x_\alpha$ are smooth,

(3) the family $\{(U_\alpha, x_\alpha\}$ is maximal relative to the conditions (1) and (2).

The pair (U_α, x_α) (or simply the mapping x_α) with $p \in x_\alpha(U_\alpha)$ is called a parametrization, or system of coordinates, of M at p. And $x_\alpha(U_\alpha)$ is called a coordinate neighborhood at p.

Remark 1. *A set M and a family of injective mappings $x_\alpha : U_\alpha \subseteq \mathbb{R}^n \longrightarrow M$ satisfying only (1) and (2) is usually called a smooth structure. In fact, a smooth manifold is generally given from a smooth structure. For that, observe that this family x_α can obviously be completed to a maximal one, by taking the union of all the parametrizations that, together with any of the previous parametrizations satisfy (2).*

Example 1. $\mathbb{R}^n$ *is a manifold of dimension n.*

Example 2. *The regular surfaces of $\mathbb{R}^3$ are manifolds of dimension 2.*

Example 3. *Let $F : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a smooth mapping of an open set U of $\mathbb{R}^n$. A point $p \in U$ is defined to be a critical point of F if the differential $dF_p : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is not surjective. The image $F(p)$ of a critical point is called a critical value of F. If a point $q \in \mathbb{R}^m$ is not a critical value, then is said to be a regular value of F. With these definitions we have that if q is a regular value of F then $F^{-1}(q)$ is empty or a manifold of dimension $n - m$.*

If we fix $c \in \mathbb{R}^n$, $r > 0$, and consider $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $F(p) = \langle p - c, p - c\rangle$ then we can obtain that $F^{-1}(r^2) =$"the sphere centered at c with radius r" is a manifold of dimension $n - 1$.

Lemma 1. *A smooth manifold M has a natural topology given in the following way: $A \subseteq M$ is an open set if and only if $x_\alpha^{-1}(A \cap x_\alpha(U_\alpha))$ is an open set in $\mathbb{R}^n$ for all α.*
With this topology, the mappings $x_\alpha : U_\alpha \longrightarrow x_\alpha(U_\alpha)$ are homeomorphisms.

In general, it is always assumed that the previous topology of a smooth manifold M satisfies the *countable basis axiom*: M can be covered by a countable number of coordinate neighborhoods.

Definition 2. *Let M_1^n and M_2^m be smooth manifolds. A mapping $\varphi : M_1 \longrightarrow M_2$ is smooth at $p \in M_1$ if given a parametrization $y : V \subseteq \mathbb{R}^m \longrightarrow M_2$ at $\varphi(p)$ there exists a parametrization $x : U \subseteq \mathbb{R}^n \longrightarrow M_1$ at p such that $\varphi(x(U)) \subseteq y(V)$ and the mapping $y^{-1} \circ \varphi \circ x : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is smooth at $x^{-1}(p)$.*

Definition 3. *Let M be a smooth manifold. A smooth function $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ is called a smooth curve in M. Suppose that $\alpha(0) = p \in M$, and let $\mathcal{D}$ be the set of functions on M that are smooth at p. The tangent vector to the curve α at $t = 0$ is a function $\alpha'(0) : \mathcal{D} \longrightarrow \mathbb{R}$ given by*

$$(\alpha'(0))(f) = \left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0}, \qquad f \in \mathcal{D}.$$

A tangent vector at p is the tangent vector at $t = 0$ of some curve $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ with $\alpha(0) = p$. The set of all tangent vectors to M at p will be denoted by T_pM.

Lemma 2. *T_pM is a vector space of dimension n.*

Proposition 1. *Let M_1 and M_2 be smooth manifolds and let $\varphi : M_1 \longrightarrow M_2$ be a smooth mapping. For every $p \in M_1$ and for each $v \in T_pM_1$, choose a smooth curve $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M_1$ with $\alpha(0) = p$, $\alpha'(0) = v$. Take $\beta = \varphi \circ \alpha$. The mapping $d\varphi_p : T_pM_1 \longrightarrow T_{\varphi(p)}M_2$ given by $d\varphi_p(v) = \beta'(0)$ is a linear mapping that does not depend on the choice of α.*

Definition 4. *The previous linear mapping $d\varphi_p$ is called the differential of φ at p.*

Definition 5. *Let M_1 and M_2 be smooth manifolds. A mapping $\varphi : M_1 \longrightarrow M_2$ is a diffeomorphism if it is smooth, bijective, and its inverse φ^{-1} is smooth. φ is said to be a local diffeomorphism at $p \in M$ if there exist neighborhoods U at p and V at $\varphi(p)$ such that $\varphi : U \longrightarrow V$ is a diffeomorphism.*

Theorem 1. *Let $\varphi : M_1 \longrightarrow M_2$ be a smooth mapping and $p \in M_1$. Then, φ is a local diffeomorphism at p if and only if $d\varphi_p : T_pM_1 \longrightarrow T_{\varphi(p)}M_2$ is an isomorphism.*
In such a case, the dimensions of M_1 and M_2 agree.

Definition 6. *Let M and N be smooth manifolds. A smooth mapping $\varphi : M \longrightarrow N$ is said to be an immersion if $d\varphi_p : T_pM \longrightarrow T_{\varphi(p)}N$ is injective for all $p \in M$. If, in addition, $\varphi : M \longrightarrow \varphi(M) \subseteq N$ is a homeomorphism, we say that φ is an embedding.*

Moreover, if $M \subseteq N$ and the inclusion $i : M \longrightarrow N$ is an embedding, we say that M is a submanifold of N.

Example 4. *The mapping $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $\varphi(x, y) = (\cos x, \sin x, y)$ is an immersion, but not an embedding.*

Example 5. *The unit sphere $\mathbb{S}^n$ is a submanifold of $\mathbb{R}^{n+1}$.*

Proposition 2. *Let $\varphi : M \longrightarrow N$ be an immersion. Then, for every point $p \in M$ there exists a neighborhood $V \subseteq M$ such that the restriction $\varphi : V \longrightarrow N$ is an embedding.*

Definition 7. *Let M be a smooth manifold. We say that M is orientable if M admits a smooth structure $\{(U_\alpha, x_\alpha)\}$ such that: for every pair α, β, with non empty intersection $x_\alpha(U_\alpha) \cap x_\beta(U_\beta)$, the differential of the change of coordinates $x_\beta^{-1} \circ x_\alpha$ has positive determinant.*

Definition 8. *A vector field X on a smooth manifold M is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_pM$. The field is smooth at $p \in M$ if X is written in a parametrization $x = (x_1, \ldots, x_n)$ of M at p as*

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}, \tag{1}$$

where each real function a_i is smooth at p.

One can think of a vector field as a mapping $X : \mathcal{D} \longrightarrow \mathcal{F}$ from $\mathcal{D} = \{\text{smooth functions on M}\}$ to $\mathcal{F} = \{\text{functions on M}\}$, locally defined from (1) as

$$X(f) = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}.$$

In particular, X is smooth if and only if $X(f)$ is a smooth function, for each smooth function f.

The set of smooth vector fields will be denoted by $\mathfrak{X}(M)$.

Example 6. *The vector field X in $\mathbb{R}^n$ defined as $X(x) = (1, 0, \ldots, 0) \equiv \frac{\partial}{\partial x_1}$ is smooth and $X(f) = \frac{\partial f}{\partial x_1}$ for every smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.*

Lemma 3. *Let X, Y be two smooth vector fields on M. Then, there exists a unique vector field Z such that, for all $f \in \mathcal{D}$, $Z(f) = (XY - YX)(f)$.*

The previous smooth vector field Z is called the (Lie) bracket $[X, Y] = XY - YX$ of X and Y. The bracket operation has the following properties:

1. Anticommutativity: $[X, Y] = -[Y, X]$, for all $X, Y \in \mathfrak{X}(M)$.

2. Linearity: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$, for all $X, Y, Z \in \mathfrak{X}(M)$ and for all $a, b \in \mathbb{R}$.

3. $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$, for all $X, Y, Z \in \mathfrak{X}(M)$ and for all $f, g \in \mathcal{D}$.

Example 7. *In $\mathbb{R}^n$ we have that $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$. Thus, for general smooth vector fields $X = (a_1, \ldots, a_n) \equiv \sum a_i \frac{\partial}{\partial x_i}, Y = (b_1, \ldots, b_n) \equiv \sum b_i \frac{\partial}{\partial x_i}$, we have*

$$[X, Y] = (X(b_1) - Y(a_1), \ldots, X(b_n) - Y(a_n)).$$

1.2 Riemannian manifolds.

Definition 9. *A Riemannian metric on a smooth manifold M is a correspondence with associates to each point $p \in M$ an inner product $\langle , \rangle_p$ (that is, a symmetric, bilinear, positive definite form) on the tangent space T_pM, which varies smoothly in the following sense:*

If $x : U \subseteq \mathbb{R}^n \longrightarrow M$ is a system of coordinates then $g_{ij} := \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$ are smooth functions for all i, j.

A smooth manifold with a given Riemannian metric is called a Riemannian manifold.

Another way to express the smoothness of the Riemannian metric is to say that $\langle X, Y \rangle$ is a smooth function for every $X, Y \in \mathfrak{X}(M)$.

The natural concept of equivalence between Riemannian manifolds is the following:

Definition 10. *Let M, N be two Riemannian manifolds. A diffeomorphism $f : M \longrightarrow N$ is called an isometry if*

$$\langle df_p(v), df_p(w) \rangle_{f(p)} = \langle v, w \rangle_p, \qquad \text{for all } p \in M, v, w \in T_pM.$$

Example 8. *The mapping $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $\varphi(x, y) = (\cos x, \sin x, y)$ is not an isometry, but it is an isometry when restricted to $(t_0, t_0 + 2\pi) \times \mathbb{R} \subseteq \mathbb{R}^2$ onto its image, with $t_0 \in \mathbb{R}$.*

Definition 11. *Let $f : M \longrightarrow N$ be an immersion (resp. embedding) between two Riemannian manifolds. We say that f is an isometric immersion (resp. isometric embedding) if*

$$\langle df_p(v), df_p(w) \rangle_{f(p)} = \langle v, w \rangle_p, \qquad \text{for all } p \in M, v, w \in T_pM.$$

Moreover, if $M \subseteq N$ and the inclusion $i : M \longrightarrow N$ is an isometric embedding, we say that M is a Riemannian submanifold of N.

Example 9. *The previous example determines an isometric immersion from $\mathbb{R}^2$ into $\mathbb{R}^3$.*

Example 10. *Let M be a smooth manifold, N be a Riemannian manifold and $f : M \longrightarrow N$ be an immersion. Then, we can define a Riemannian metric on M in such a way that f is an isometric immersion. For that, it is enough to define:*

$$\langle v, w \rangle_p := \langle df_p(v), df_p(w) \rangle_{f(p)}, \qquad \text{for all } p \in M, v, w \in T_pM.$$

In this sense, regular surfaces in $\mathbb{R}^3$ are considered as Riemannian submanifolds of $\mathbb{R}^3$. The same is true for curves in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition 12. *Let $I \subseteq \mathbb{R}$ be an open interval, and $\alpha : I \longrightarrow M$ be a smooth curve into a Riemannian manifold. The restriction of α to a closed interval $[a, b] \subseteq I$ is called a segment, and its length is defined by*

$$L_a^b(\alpha) = \int_a^b \sqrt{\langle \alpha'(t), \alpha'(t) \rangle}\, dt.$$

Theorem 2. *Let $\varphi : M \longrightarrow N$ be a smooth mapping between two Riemannian manifolds. Then, φ is an isometric immersion if and only if φ preserves the length of every smooth curve (that is, $L_a^b(\alpha) = L_a^b(\varphi \circ \alpha)$ for all smooth curve α in M).*

Lemma 4. *Let M be a Riemannian manifold and $X, Y \in \mathfrak{X}(M)$. Then, there exists a new smooth vector field $\nabla_X Y$ given by*

$$\begin{aligned} 2\langle \nabla_X Y, Z \rangle &= X(\langle Y, Z \rangle) + Y(\langle X, Z \rangle) - Z(\langle X, Y \rangle) \\ &\quad -\langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle + \langle [X, Y], Z \rangle \end{aligned}$$

for all $Z \in \mathfrak{X}(M)$.

Definition 13. *The mapping $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ which associates (X, Y) to the previous smooth vector field $\nabla_X Y$ is called the Levi-Civita (or Riemannian) connection of the Riemannian manifold M.*

This mapping satisfies the following properties:

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$,
3. $\nabla_X (fY) = f\nabla_X Y + X(f)Y$,
4. $X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$,
5. $\nabla_X Y - \nabla_Y X = [X, Y]$,

for all $X, Y, Z \in \mathfrak{X}(M)$, $f, g \in \mathcal{D}$.

Example 11. *In $\mathbb{R}^n$ we obtain that $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$. Thus, for general $X, Y \in \mathfrak{X}(\mathbb{R}^n)$ one has $\nabla_X Y = (X(b_1), \ldots, X(b_n))$, where $Y = (b_1, \ldots, b_n)$.*

Example 12. *Let M a Riemannian submanifold of the Riemannian manifold N. If we denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and N, respectively, then*

$$\nabla_X Y = (\overline{\nabla}_{\overline{X}} \overline{Y})^T$$

where $\overline{X}, \overline{Y}$ are smooth vector fields on N which restriction on M agrees with X, Y, and $(\cdot)^T$ denotes the tangent part of the vector field on M.

Thus, the Levi-Civita connection of the sphere $\mathbb{S}^n$ with its standard metric as a Riemannian submanifold of $\mathbb{R}^{n+1}$ is given by

$$(\nabla_X Y)(p) = (\overline{\nabla}_X Y)(p) - \langle \overline{\nabla}_X Y, p \rangle p,$$

where $\overline{\nabla}$ is the Levi-Civita connection of $\mathbb{R}^{n+1}$, which is given in the previous example.

Definition 14. *Let M be a Riemannian manifold with Levi-Civita connection ∇. We define the (Riemann) curvature tensor R on M as*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

The Riemann curvature tensor is linear in each component, that is,

1. $R(fX_1 + gX_2, Y)Z = fR(X_1, Y)Z + gR(X_2, Y)Z$,
2. $R(X, fY_1 + gY_2)Z = fR(X, Y_1)Z + gR(X, Y_2)Z$,
3. $R(X, Y)fZ_1 + gZ_2 = fR(X, Y)Z_1 + gR(X, Y)Z_2$,

for all $f, g \in \mathcal{D}$.

Associated with the previous Riemann curvature tensor, we also consider the new tensor that will be also denoted by R:

$$R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle, \qquad \text{for all } X, Y, Z, W \in \mathfrak{X}(M).$$

Observe that this new tensor is linear in the four components, and associates a smooth function on M to the four smooth vector fields. Some elementary properties of this tensor are the following:

1. $R(X, Y, Z, W) = -R(Y, X, Z, W)$,

2. $R(X, Y, Z, W) = -R(X, Y, W, Z)$,

3. $R(X, Y, Z, W) = R(W, Y, Z, X)$.

The linearity of R in each argument allows to define R for vectors, and not only for smooth vector fields. Thus, given $x, y, z, w \in T_pM$ and $X, Y, Z, W \in \mathfrak{X}(M)$ such that $X(p) = x, Y(p) = y, Z(p) = z, W(p) = w$ then the definitions

$$R(x, y)z := (R(X, Y)Z)(p), \qquad R(x, y, z, w) := R(X, Y, Z, W)(p)$$

do not depend on X, Y, Z, W.

Example 13. *For $\mathbb{R}^n$ the Riemann curvature tensor $R \equiv 0$.*

Example 14. *In $\mathbb{S}^n$ the Riemann curvature tensor satisfies $R(X, Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$.*

Definition 15. *Let M be a Riemannian manifold, $p \in M$ and π be a 2-dimensional subspace of the tangent space T_pM. We define the sectional curvature of π as the number*

$$K(\pi) = \frac{R(v, w, w, v)}{\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2},$$

where $\{v, w\}$ is any basis of $\pi \subseteq T_pM$.

It is important to remark that the previous number $K(\pi)$ does not depend on the chosen basis $\{v, w\}$.

Certain combinations of sectional curvature appear with such frequency that they deserve special names:

Definition 16. *Let M be a Riemannian manifold, and $p \in M$. Given a unit vector $v \in T_pM$ we define the Ricci curvature in the direction v as the average:*

$$Ric_p(v) = \frac{1}{n-1}\sum_{i=1}^{n-1} K(Lin\,\{v, e_i\}),$$

where $\{e_1, \ldots, e_{n-1}, v\}$ is an orthonormal basis of T_pM, and $Lin\,\{v, e_i\}$ denotes the plane generated by v and e_i.

Moreover, we define the scalar curvature of M at p as the average:

$$\rho(p) = \frac{1}{n}\sum_{i=1}^{n} Ric_p(e_i),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM.

Again, these definitions do not depend on the chosen basis.

Example 15. *In $\mathbb{R}^n$ we have $K \equiv 0$, $Ric_p \equiv 0$, and $\rho \equiv 0$.*

Example 16. *In $\mathbb{S}^n$ we have $K \equiv 1$, $Ric_p \equiv 1$, and $\rho \equiv 1$.*

2 The second fundamental form and principal curvatures.

In this section we will consider isometric immersions $f : M \longrightarrow \overline{M}$ between Riemannian manifolds. Since, given a point $p \in M$ there exists a neighborhood $U \subseteq M$ of p such that $f : U \longrightarrow f(U)$ is an isometry, we will identify U with $f(U)$ and consider $f(U)$ as a Riemannian submanifold of $\overline{M}$.

Thus, for every $p \in M$, the inner product on $T_p\overline{M}$ splits $T_p\overline{M}$ into the direct sum

$$T_p\overline{M} = T_pM \oplus (T_pM)^{\perp}.$$

In this way, if we denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections in M and $\overline{M}$, we have

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \nabla_X Y + II(X,Y),$$

for any $X, Y \in \mathfrak{X}(M)$, where $\overline{X}, \overline{Y}$ are local extensions to $\overline{M}$ (see Example 12).

The vector $II(X,Y)$ does not depend on the extensions $\overline{X}, \overline{Y}$ of the smooth vector fields X, Y.

Definition 17. *The mapping $II : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)^{\perp}$ given by*

$$II(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y$$

is called the second fundamental form of the isometric immersion. Here, $\mathfrak{X}(M)^{\perp}$ denotes the perpendicular smooth vector fields.

Lemma 5. *The second fundamental form is bilinear and symmetric. That is,*

1. *$II(fX + gY, Z) = fII(X,Z) + gII(Y,Z)$, for all $f, g \in \mathcal{D}$.*
2. *$II(X, fY + gZ) = fII(X,Y) + gII(X,Z)$, for all $f, g \in \mathcal{D}$.*
3. *$II(X,Y) = II(Y,X)$,*

Since II is bilinear, we can define II for tangent vectors and not only for smooth vector fields. Thus, given $v, w \in T_pM$ then

$$II(v,w) := II(X,Y)$$

where X, Y are any smooth vector fields such that $X(p) = v, Y(p) = w$.

Definition 18. *Let $f : M \longrightarrow \overline{M}$ be an isometric immersion, $p \in M$ and $\eta \in (T_pM)^{\perp}$. Then, the symmetric bilinear form*

$$II_\eta(v,w) = \langle II(v,w), \eta\rangle, \qquad v, w \in T_pM,$$

or its associated quadratic form

$$II_\eta(v) = \langle II(v,v), \eta\rangle, \qquad v \in T_pM,$$

are called the second fundamental form of f at p along the normal vector η.

Since, II_η is a bilinear symmetric form, it is associated to a linear self-adjoint operator $S_\eta : T_pM \longrightarrow T_pM$, which is defined by

$$\langle S_\eta(v), w\rangle = II_\eta(v,w) = \langle II(v,w), \eta\rangle, \qquad v, w \in T_pM.$$

Moreover, since S_η is a linear self-adjoint operator, S_η is diagonalizable.

Definition 19. *S_η is called the shape operator along the normal vector η, its eigenvalues the principal curvatures along η, and its eigenvectors the principal directions along η.*

Example 17. *Consider the isometric immersion given by the inclusion $i : \mathbb{S}^n \longrightarrow \mathbb{R}^{n+1}$. Then, for every point $p \in \mathbb{S}^n$, $(T_pM)^\perp = Lin\{p\}$. Moreover, from Example 12,*

$$II(X,Y) = \langle \overline{\nabla}_X Y, p\rangle p = -\langle X, Y\rangle p.$$

Thus, if we consider $\eta = -p$ as a normal vector, then $\langle S_\eta(v), w\rangle = \langle v, w\rangle$, that is,

$$S_\eta(v) = v.$$

Example 18. *Consider the isometric immersion $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ given by $f(x,y) = (\cos x, \sin x, y, 0)$, then for $p = (x_0, y_0)$ we have $(T_p\mathbb{R}^2)^\perp = Lin\{(\cos x_0, \sin x_0, 0, 0), (0,0,0,1)\}$.*

Then one has

$$II\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = -(\cos x, \sin x, 0, 0), \quad II\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0, \quad II\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = 0.$$

For instance, for $\eta = (0,0,0,1)$ one gets $S_\eta(v) = 0$.

A specially interesting case happens when the isometric immersion $f : M \longrightarrow \overline{M}$ satisfies that $dim(\overline{M}) - dim(M) = 1$. In such a case the immersion is called a hypersurface.

For a hypersurface $dim((T_pM)^\perp) = 1$ and there is a unique unit vector $\eta \in (T_pM)^\perp$, up to sign. The shape operator S_η is then denoted simply by S and called the shape operator of the hypersurface. The same happens for its principal curvatures and principal directions. Observe that S and the principal curvatures change sign if we replace η by $-\eta$. However, the principal directions agree.

Definition 20. *Let $f : M \longrightarrow \overline{M}$ be a hypersurface. If we denote by $k_1, \ldots, k_n$ the principal curvatures of S then the quantities*

$$H = \frac{1}{n} trace(S) = \frac{k_1 + \ldots + k_n}{n}, \qquad K_e = det(S) = k_1 \ldots k_n$$

are called the mean curvature and the Gauss-Kronecker curvature of the immersion.

Example 19. *Let us consider M as the graph in $\mathbb{R}^3$ parametrized by $\psi(x,y) = (x, y, f(x,y))$ then its mean curvature and Gauss-Kronecker curvature are given by*

$$H = \frac{(1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy}}{2(1+f_x^2+f_y^2)^{3/2}}, \qquad K_e = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1+f_x^2+f_y^2)^2}.$$

3 Fundamental equations.

Let $f : M \longrightarrow \overline{M}$ be an isometric immersion, $X \in \mathfrak{X}(M)$ and $\eta \in \mathfrak{X}(M)^{\perp}$, then the tangent part of $\overline{\nabla}_X \eta$ is given by $-S_\eta(X)$.

Thus, using the decomposition $T_p\overline{M} = T_pM \oplus (T_pM)^{\perp}$ we write

$$\overline{\nabla}_X \eta = -S_\eta(X) + \nabla^{\perp}_X \eta.$$

$\nabla^{\perp}$ is called the normal connection of the immersion and satisfies:

1. $\nabla^{\perp}_{fX_1+gX_2}\eta = f\nabla^{\perp}_{X_1}\eta + g\nabla^{\perp}_{X_2}\eta$, $f, g \in \mathcal{D}$,
2. $\nabla^{\perp}_X(\eta_1 + \eta_2) = \nabla^{\perp}_X\eta_1 + \nabla^{\perp}_X\eta_2$,
3. $\nabla^{\perp}_X(f\eta) = f\nabla^{\perp}_X\eta + X(f)\eta$, $f \in \mathcal{D}$.

Definition 21. *The normal curvature $R^{\perp}$ of the isometric immersion is defined by*

$$R^{\perp}(X,Y)\eta = \nabla^{\perp}_X\nabla^{\perp}_Y\eta - \nabla^{\perp}_Y\nabla^{\perp}_X\eta - \nabla^{\perp}_{[X,Y]}\eta.$$

Now, we deduce different equations that an isometric immersion must satisfy and generalize the classical equations of Gauss and Codazzi in the theory of surfaces.

Theorem 3 (Gauss equation)**.** *Let $f : M \longrightarrow \overline{M}$ be an isometric immersion, then*

$$R(X,Y,Z,W) = R(X,Y,Z,W) + \langle II(X,Z), II(Y,W)\rangle - \langle II(X,W), II(Y,Z)\rangle.$$

Theorem 4 (Codazzi equation)**.** *Let $f : M \longrightarrow \overline{M}$ be an isometric immersion, then*

$$\overline{R}(X,Y,Z,\eta) = (\overline{\nabla}_X II)(Y,Z,\eta) - (\overline{\nabla}_Y II)(X,Z,\eta),$$

where $(\overline{\nabla}_X II)(Y,Z,\eta) = X(II(Y,Z,\eta)) - II(\nabla_X Y, Z, \eta) - II(Y, \nabla_X Z, \eta) - II(Y,Z,\nabla^{\perp}_X\eta)$ and $II(X,Y,\eta) := \langle II(X,Y), \eta\rangle$.

Theorem 5 (Ricci equation)**.** *Let $f : M \longrightarrow \overline{M}$ be an isometric immersion, then*

$$\overline{R}(X,Y,\eta,\zeta) = \langle R^{\perp}(X,Y)\eta, \zeta\rangle - \langle [S_\eta, S_\zeta]X, Y\rangle,$$

where $[S_\eta, S_\zeta] = S_\eta \circ S_\zeta - S_\zeta \circ S_\eta$.

Example 20. *Let $f : M \longrightarrow \mathbb{R}^3$ be an isometric immersion, then its sectional (or intrinsic) curvature agrees with its Gauss (or extrinsic) curvature, that is,*

$$K(p) := K(T_pM) = k_1k_2,$$

where k_1, k_2 are the principal curvatures at $p \in M$. This is the classical Gauss's Theorema Egregium.